"BABEŞ-BOLYAI" UNIVERSITY
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ON THE CONVERGENCY OF A STEFFENSEN-TYPE METHOD

ION PĂVĂLOIU

1. In the paper [1] I.K. Argyros adopts for the divided difference of the mapping $f : X_1 \to X_2$, where X_1 and X_2 are Banach spaces, the following definition:

Definition 1. *One calls divided difference of the application f at the points $x, y, \in X_1$ a linear application $[x, y; f] \in \mathcal{L}(X_1, X_2)$ which fulfils the following conditions:*

(a) $[x, y; f](y - x) = f(y) - f(x)$ *for every* $x, y \in D \subseteq X_1$;
(b) *there exist the real constants* $l_1 > 0,\ l_2 > 0,\ l_3 > 0,\ p \in (0, 1]$ *such that for every* $x, y, u \in D$ *the following inequality holds:*

$$\|[y, u; f] - [x, y; f]\| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_3 \|u - y\|^p .$$

In [4] there are obtained refinements of Argyros results concerning the secant method applied to the solution of the equation:

$$f(x) = 0 \tag{1}$$

where $f : X_1 \to X_2$

2. We shall study further down the convergence of Steffensen's method for the solution of equation (1), namely the convergence of the sequence $(x_n)_{n\geq 0}$ generated by means of the following procedure:

$$(2)\qquad x_{n+1}=x_n-[x_n,g(x_n);f]^{-1}f(x_n),\qquad x_0\in x_1,\ n=0,1,\ldots,$$

where $g:X_1\to X_1$ is an operator having at least one fixed point which coincides with the solution of equation (1).

Obviously, the sequence $(x_n)_{n\geq 0}$ can be generated by means of the procedure (2) if at each iteration step there exists the mapping $[x_n,g(x_n);f]^{-1}$.

For our purpose observe firstly that the following identities:

$$(3)\qquad x_n-[x_n,g(x_n);f]^{-1}f(x_n)=$$

$$\begin{aligned}(4)\qquad &=g(x_n)-[x_n,g(x_n);f]^{-1}f(g(x_n))f(x_{n+1})\\ &=f(g(x_n))+[x_n,g(x_n);f](x_{n+1}-g(x_n))\\ &\quad+([g(x_n),x_{n+1};f]-[x_n,g(x_n);f])\ (x_{n+1}-g(x_n))\end{aligned}$$

hold for every $n=0,1,\ldots$

Let $x_0\in X_1$ be an element, and consider the nonnegative real numbers: $B,\varepsilon_0,\rho_0,p\in(0,1],\ \alpha,\beta,q\geq 1,\ l_1,l_2$ and l_3, where

$$\rho_0=\beta\alpha\big(l_1B^p+l_2B^p+l_3B^p\alpha^p\left\|f(x_0)\right\|^{p(q-1)}\big)$$

and

$$\varepsilon_0=\rho^{1/(p+q-1)}\left\|f(x_0)\right\|.$$

Denote $r=\max\{B,\beta\}$ and suppose that $S\subseteq D$, where:

$$S=\Big\{x\in X_1:\|x-x_0\|\leq\tfrac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}(1-\varepsilon_0^{p+q-1})}\Big\}.$$

The following theorem holds:

Theorem 1. *If the constants $B,\ \varepsilon_0,\ \rho_0,p,\alpha,\beta,q,l_1,l_2,l_3$, the mapping f and g, and the initial element $x_0\in X_1$, as well, fulfil the conditions:*

(I) *for every $x,y\in S$ there exists $[x,y;f]^{-1}$, and* $\left\|[x,y;f]^{-1}\right\|\leq B$

(II) *for every $x\in S$, $\|f(g(x))\|\leq\alpha\|f(x)\|^q$;*

(III) *for every $x\in S$, $\|x-g(x)\|\leq\beta\|f(x)\|$;*

(IV) *the divided difference of the mapping f fulfils the conditions* (a) *and* (b) *specified in the definition given in Section* 1;

(V) $\varepsilon_0 < 1$,

then the sequence $(x_n)_{n\geq 0}$ *generated by the procedure* (2) *is convergent, and, if we denote* $\bar{x} = \lim x_n$, *then* $f(\bar{x}) = 0$ *and the following delimitation holds:*

$$\|x - x_n\| \leq \frac{r\rho_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)}.$$

Proof. Consider $x_0 \in X_1$ for which the condition (V) is fulfilled. Taking into account the condition (b) and the procedure (2), from the identities (3) and (4) it results:

$$\|x_1 - x_0\| \leq B\|f(x_0)\| \leq \frac{B\varepsilon_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}}\|f(x_0)\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},$$

from which follows $x_1 \in S$.

Here was used the inequality:

$$\|g(x_0) - x_0\| \leq \beta\|f(x_0)\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},$$

from which follows that $g(x_0) \in S$.

Now, considering the above results, we have:

$$\begin{aligned}\|f(x_1)\| &\leq \|[g(x_0), x_1; f] - [x_0, g(x_0); f]\| \cdot \|x_1 - g(x_0)\| \\ &\leq \beta\alpha\left[l_1B^p + l_2B^p + l_3B^p\alpha^p\|f(x_0)\|^{p(q-1)}\right]\|f(x_0)\|^{p+q} \\ &= \rho_0\|f(x_0)\|^{p+q}\end{aligned}$$

This inequality leads to:

$$\rho_0^{1/(p+q-1)}\|f(x_1)\| \leq \rho_0^{1/(p+q-1)}\|f(x_0)\|^{p+q}$$

or, using the notation $\varepsilon_1 = \rho_0^{1/(p+q-1)}\|f(x_1)\|$:

$$\varepsilon_1 \leq \varepsilon_0^{p+q}$$

From this inequality follows that $\|f(x_1)\| \leq \|f(x_0)\|$, and if

$$\rho_1 = \beta\alpha\left(l_1B^p + l_2B^p + l_3B^p\alpha^p\|f(x_1)\|^{p(q-1)}\right)$$

then $\rho_1 \leq \rho_0$.

Suppose now that the following properties hold:

(α) $x_p \in S$;
(β) $\|f(x_p)\| \leq \|f(x_{p-1})\|$;
(γ) $\varepsilon_p \leq \varepsilon_0^{(p+q)^p}$, $\varepsilon_p = \rho_0^{1/(p+q-1)} \|f(x_p)\|$, $p = 1, 2, \ldots, k$

From (2) for $n = k$ we obtain:

$$\|x_{k+1} - x_k\| \leq B \|f(x_k)\| \leq \frac{r\varepsilon_k}{\rho_0^{1/(p+q-1)}} \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}},$$

which leads to:

$$\begin{aligned}\|x_{k+1} - x_0\| &\leq \frac{r}{\rho_0^{1/(p+q-1)}} \left(\varepsilon_0 + \varepsilon_0^{p+q} + \varepsilon_0^{(p+q)^2} + \ldots + \varepsilon_0^{(p+q)^k}\right) \\ &\leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},\end{aligned}$$

namely $x_{k+1} \in S$.

Here was used the inequality:

$$\|g(x_k) - x_k\| \leq B \|f(x_k)\| \leq \frac{r\rho_0^{1/(p+q-1)}}{\rho_0^{1/(p+q-1)}} \|f(x_k)\| \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{1/(p+q-1)}},$$

from which follows immediately:

$$\|g(x_k) - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},$$

that is, $g(x_k) \in S$.

As to $\|f(x_{k+1})\|$ we have:

$$\|f(x_{k+1})\| \leq \beta\alpha \left(l_1 B^p + l_2 B^p + l_3 \alpha^p B^p \|f(x_k)\|^{p(q-1)}\right) \|f(x_k)\|^{p+q}$$

namely

$$\|f(x_{k+1})\| \leq \rho_0 \|f(x_k)\|^{p+q},$$

which yields:

$$\varepsilon_{k+1} \leq \varepsilon_k^{p+q} \leq \varepsilon_0^{(p+q)^{k+1}}.$$

By virtue of the above proved results follows that the properties (α)–(γ) hold for every $p \in \mathbb{N}$.

We prove further down that the sequence $(x_n)_{n\geq 0}$ is a fundamental sequence. Indeed, we have:

$$\begin{aligned}
\|x_{n+s}-x_n\| &\leq \|x_{n+s}-x_{n+s-1}\| + \|x_{n+s-1}-x_{n+s-2}\| + \ldots + \|x_{n+1}-x_n\| \\
&\leq B\left(\|f(x_n)\| + \|f(x_{n+1})\| + \ldots + \|f(x_{n+s-1})\|\right) \\
&\leq \tfrac{B}{\rho_0^{1/(p+q-1)}}\left(\varepsilon_0^{(p+q)^n} + \varepsilon_0^{(p+q)^{n+1}} + \ldots + \varepsilon_0^{(p+q)^{n+s-1}}\right) \\
&\leq \tfrac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}}\left(1 + \varepsilon^{p+q-1} + \varepsilon^{(p+q)^2-1} + \varepsilon^{(p+q)^{s-1}-1}\right) \\
&\leq \tfrac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},
\end{aligned}$$

that is, for every $s, n \in \mathbb{N}$ the following inequality holds:

$$\|x_{n+s}-x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},$$

from which, since $\varepsilon_0 < 1$, it results that the sequence $(x_n)_{n\geq 0}$ is fundamental. Since X_1 is a Banach space, there exists $\lim\limits_{n\to\infty} x_n = \bar{x}$,

and

$$\|\bar{x}-x_n\| \leq \tfrac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{1/(p+q-1)}\left(1-\varepsilon_0^{p+q-1}\right)},$$

which leads, for $n = 0$, to $\bar{x} \in S$.

From the inequality $\varepsilon_n \leq \varepsilon_0^{(p+q)^n}$, for $n \to \infty$, we obtain:

$$f(\bar{x}) = \lim_{n\to\infty} f(x_n) = 0,$$

and one sees that $\bar{x}$ is the solution of the equation (1). $\square$

References

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available soon, refresh and click here → [3] Păvăloiu, I., *Introduction in the Theory of Approximation of Equations Solutions,* Dacia Ed., Cluj-Napoca 1976 (in Romanian).

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Institutul de Calcul
Oficiul Poştal 1
C.P. 68
3400 Cluj-Napoca
Romania

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