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REMARKS ON THE SECANT METHOD FOR THE SOLUTION OF NONLINEAR OPERATORIAL EQUATIONS

Ion Păvăloiu

This note has for purpose some refinements of the convergence conditions and error delimitations obtained by I.K. Argyros in [1] with respect to the secant method for the solution of the equation:

$$f(x) = 0, \tag{1}$$

where $f : x_1 \to x_2$ is a nonlinear operator, while x_1 and x_2 are Banach spaces.

If we denote by $[x, y; f]$ the divided difference of the mapping f on the point x and y, then for fixed x, y we have $[x, y; f] \in \mathcal{L}(X_1, X_2)$. It is known that in certain conditions the sequence $(x_n)_{n\geq 0}$ generated by the secant method:

$$x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} fx_n),\ x_0, x_1 \in x_1,\ n = 1, 2, \ldots \tag{2}$$

converges to the solution x^* of equation (1).

1. Generalizing a result on J.E. Dennis [2], I.K. Argyros [1] studies the convergence of the method (2) with the assumptions that the operator f is Fréchet differentiable, while the derivative $f'(x)$ fulfils a Hölder-like condition on a set $D \subset X_1$,namely there exist a constant $c > 0$ and number $p \in (0, 1]$ such that the inequality:

$$\left\| f'(x) - f'(y) \right\| \leq C \left\| x - y \right\|^p \tag{3}$$

holds for every $x, y \in D$. In this case we shall say that $f'(\cdot) \in H_D(C, p)$.

In the quoted paper I.K. Argyros defines the divided difference operator $[x, y; f]$ as a linear operator which fulfils the conditions:

$$[x, y; f](y - x) = f(y) - f(x), \qquad \forall x, y \in D, \tag{4}$$

and, in addition, for every $x, y, u \in D$ the following inequality holds:

$$\|[x, y; f] - [y, u; f]\| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_2 \|y - u\|^p, \tag{5}$$

where $l_1 \geq 0,\ l_2 \geq 0$ are constants which do not depend on x, y and u, while $p \in (0, 1]$.

Let x^* be a simple solution of (1). We mean by that the mapping $f'(x^*)$ admits a bounded inverse mapping, and if $[x^*, x^*; f] = f'(x^*)$ then $[x^*, x^*; f]$ admits a bounded inverse mapping. Thus the continuousness of the mapping $[x, y; f]$ with respect to the variable x and y ensures the existence of a number $\varepsilon > 0$ such that the mapping $[x, y; f]$ admits a bounded inverse mapping for every $x, y \in U(x^*, \varepsilon)$, where $U(x^*, \varepsilon) = \{x \in X_1 : \|x - x^*\| < \varepsilon\}$ that is, the set $B(x, y) = \|[x, y; f]^{-1}\|$ is uniformly bounded in $U(x^*, \varepsilon) = \{x \in X_1 : \|x - x^*\| \leq \varepsilon\}$.

Theorem 1. [1] *Let $f : X_1 \to X_2$ and let $D \subset X_1$ an open set. The following conditions are fulfilled:*

(a) *$x^* \in D$ is a simple solution of the equation* (1);
(b) *there exist $\varepsilon \in 0,\ b > 0$ such that $\|[x, y; f]^{-1}\| \leq b$ for every $x, y \in U(x^*, \varepsilon)$;*
(c) *there exists a convex set $D_0 \subset D$ such that $x^* \in D_0$, and there exists $\varepsilon_1 > 0$, with $0 < \varepsilon_1 < \varepsilon$ such that $f'(\cdot) \in H_{D_0}(C, p)$ for every $x, y \in D_0$ and $U(x^*, \varepsilon_1) \subset D_0$.*

Let $r>0$ such that:

(6) $$0<r<\min\{\varepsilon_1,(q(p))^{-1/p}\}$$

where:

(7) $$q(p)=\tfrac{b}{p+1}\left[2^p(l_1+l_2)(l+p)+C\right].$$

Then, if $x_0x_1\in\bar{U}(x^*,r)$, the iterates x_n, $n=2,3,\ldots,$ generated by (2) are well defined and belong to the set $\bar{U}(x^*,r)$, while the sequence $(x_n)_{n\geq 0}$ converges to the unique solution x^* of equation (1).

Moreover, the following estimation:

(8) $$\|x_{n+1}-x^*\|\leq\gamma_1\|x_{n-1}-x^*\|^p\cdot\|x_n-x^*\|+\gamma_2\|x_n-x^*\|^{p+1}$$

holds for sufficiently great n, where:

(9) $$\gamma_1=b(l_1+l_2)2^p,$$

(10) $$\gamma_2=\tfrac{bC}{1+p}$$

while l_1,l_2 and p were precised by the relation (5).

In order to prove this theorem the author uses the following two lemmas:

Lemma 1. [1]. *Let $f:X_1\to X_2$ and $D\subset X_1$. Suppose that D is an open set and $f'(\cdot)$ does exist in every point of D. If, for a convex set $D_0\subseteq D, f'(\cdot)\in H_{D_0}(C,p)$, then for every $x,y\in D_0$ the following inequality holds:*

$$\left\|f(x)-f(y)-f'(x)(y-x)\right\|\leq\tfrac{C}{1+p}\|x-y\|^{1+p}.$$

Lemma 2. [1]. *If $[x,y;f]$ fulfils the conditions (4) and (5), the following relations hold:*

(a) $[x,x;f]=f'(x)$ *for every $x\in D_0$;*
(b) $f'(\cdot)\in H_{D_0}(2(l_1+l_2),p)$.

From the proof of Theorem 1 follows, for the error estimation and for the convergence speeds of the sequence $(x_n)_{n\geq 0}$, the inequality:

$$\|x_{n+1} - x^*\| \leq (M(r))^{n+1} \|x_0 - x^*\| \tag{11}$$

where one shows that $M(r) \in (0,1)$.

2. We shall make further down some remarks upon the above exposed results, showing that the hypotheses imposed in [1] can lead to more rich conclusions with respect to both the convergency order of the secant method and the error estimation.

Suppose that x_0 and x_1 fulfil the conditions:

(a') $\|x^* - x_0\| \leq \alpha d_0$;
(b') $\|x^* - x_1\| \leq \min\{\alpha d_0^{t_1}, \|x^* - x_0\|\}$

where $0 < d_0 < 1$, $\alpha = (q(p))^{-1}$, while t_1 is the positive root of the equation:

$$t^2 - t - p = 0 \tag{12}$$

$$\text{namely } t_1 = \tfrac{1+(1+4p)^{1/2}}{2}.$$

Using the condition (4) and (5), Lemmas 1 and 2, and the hypotheses of 1, it results easily from (2),for $n = 1$, the inequality [1]:

$$\|x_2 - x^*\| \leq \gamma_1 \|x_0 - x^*\|^p \|x_1 - x^*\| + \gamma_2 \|x_1 - x^*\|^{p+1} \tag{13}$$

from which, using (a') and (b') and the fact that t_1 is a root of equation (12), we obtain:

$$\begin{aligned}
\|x_2 - x^*\| &\leq \gamma_1 \alpha^p d_0^p \alpha d_0^{t_1} + \gamma_2 \alpha^{1+p} d_0^{t_1(1+p)} \\
&= \alpha^{1+p} \left(\gamma_1 d_0^{t_1+p} + \gamma_2 d_0^{t_1(1+p)}\right) \\
&= \alpha^{1+p} d_0^{t_1+p} \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \\
&= \alpha d_0^{t_1^2} \left(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\right) \alpha^p.
\end{aligned}$$

But

$$\left(\gamma_1+\gamma_2 d_0^{p(t_1-1)}\right)\alpha^p=\frac{\gamma_1+\gamma_2 d_0^{p(t_1-1)}}{\gamma_1+\gamma_2}<1,$$

then the following inequality holds

$$\|x_2-x^*\|\le \alpha d_0^{t_1^2}.$$

We prove now that $\|x_2-x^*\|\le\|x_1-x^*\|$. From the inequality (13) we obtain:

$$\|x_2-x^*\|\left(\gamma_1\alpha^p d_0^p+\gamma_2\alpha^p d_0^{t_1p}\right)\|x_1-x^*\|\le$$

$$\le \alpha p d_0^p\left(\gamma_1+\gamma_2 d_0^{p(t_1-1)}\right)\|x_1-x^*\|<\|x_1-x^*\|$$

since $d_0^p<1$ and, as we saw above, $\alpha^p\left(\gamma_1+\gamma_2 d_0^{p(t_1-1)}\right)<1$.

Assume now that for $n\in\mathbb{N}$, $n\ge 2$, the following relations hold:

(a”) $\|x_{n-1}-x^*\|\le\alpha d_0^{t_1^{n-1}}$;

(b”) $\|x_n-x^*\|\le\min\{\alpha d_0^{t_1^n},\|x_{n-1}-x^*\|\}$

Proceeding as in the case of x_2, and taking into account (a”), (b”) and (8), we obtain:

$$\|x_{n+1}-x^*\|\le\alpha^{1+p}d_0^{t_1^{n+1}}\cdot\left(\gamma_1+\gamma_2 d_0^{pt_1^{n-1}(t_1+1)}\right)=$$

$$=\alpha d_0^{t_1^{n+1}}\cdot\alpha^p\left(\gamma_1+\gamma_2 d_0^{pt_1^{n-1}(t_1-1)}\right)\le\alpha d_0^{t_1^{n+1}},$$

since, as previously, it is easy to show that:

$$\alpha^p\left(\gamma_1+\gamma_2 d_0^{pt_1^{n-1}(t_1-1)}\right)<1$$

In order to complete the proof, we shall show that:

$$\|x_{n+1}-x^*\|\le\|x_n-x^*\|$$

Indeed, form (8) we deduce:

$$\|x_{n+1}-x^*\|\le(??\gamma_1\alpha^p d_0^{pt_1^{n-1}}+\gamma_2\alpha^p d_0^{pt_1^n})\|x_n-x^*\|.$$

But $d_0 < 1$ and $\alpha^p\big(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}\big) < 1$, therefore:

$$\|x_{n+1} - x^*\| \leq \|x_n - x^*\|.$$

We proved in this way the following theorem:

Theorem 2. *If the conditions of Theorem* 1 *are fulfilled, with the difference that* x_0 *and* x_1 *are chosen in such a manner to verify the relations* (a') *and* (b'), *where* $\alpha = (q(p))^{-1/p}$ *and* $d_0 \in (0,1)$, *then, for every* $n \in \mathbb{N}$, $x_n \in U = \{x \in X_1 | \, \|x - x^*\| < \alpha\}$ *and the following inequality holds:*

$$\|x_{n+1} - x^*\| \leq \alpha d_0^{t_1^{n+1}}, \qquad n = 0, 1, \ldots \tag{14}$$

Remark. The inequality (14) contains in its right-hand side a number substantially smaller than that yielded by relation (11).

References

[1] Argyros, I.K., *The secant method and fixed points of nonlinear operators*, Mh. Math. 106, 85–94 (1988).

[2] Dennis, J.E., *Toward a unified convergence theory for Newton like methods*, Nonlinear Functional analysis and Applications (Ed. by L.B.Rall), pp.425–472, New York, John Wiley (1986).

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[3] Păvăloiu, I., *Introduction to the Theory of Approximation of Equations Solutions*, Dacia Ed., Cluj-Napoca, 1976 (in Romanian).

Institutul de Calcul
Oficiul Postal 1
C.P. 68
3400 Cluj-Napoca
Romania

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