

OBSERVATIONS CONCERNING SOME APPROXIMATION  
METHODS FOR THE SOLUTIONS OF OPERATOR EQUATIONS

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1. INTRODUCTION

The purpose of this paper is to give some completions to some results, recently appeared in the literature, concerning the convergence and the error bounds of some methods for solving operatorial equations, when the Fréchet derivatives or the divided differences of the operators are Hölder continuous.

Let  $f : X \rightarrow Y$  be an application, where  $X$  and  $Y$  are Banach spaces. We shall define the divided difference of a certain order in the following way: let  $u_i \in X$   $i = 1, n + 1$ , where  $u_i \neq u_j$  for  $i \neq j$ .

DEFINITION 1.1. [8]. *The divided difference of the first order of the application  $f$  at  $u_k, u_s \in X$  is an application  $[u_k, u_s; f] \in \mathcal{L}(X, Y)$  which verifies:*

- a)  $[u_k, u_s; f](u_s - u_k) = f(u_s) - f(u_k)$
- b) *if  $f$  is Fréchet differentiable, at  $u_s$ , then*

$$[u_s, u_s; f] = f'(u_s).$$

*We suppose that there have been defined the applications*

$$[u_k, u_{k+1}, \dots, u_{k+m-1}; f] \in \mathcal{L}(X^{m-1}, Y) \text{ and}$$

$$[u_{k+1}, u_{k+2}, \dots, u_{k+m}; f] \in \mathcal{L}(X^{m-1}, Y),$$

*called the divided differences of the order  $m - 1$ , where  $k + m \leq n$ .*

DEFINITION 1.2. [8]. *The divided difference of the order  $m$  of the application  $f$  in  $u_k, u_{k+1}, \dots, u_{k+m} \in X$ , is an application*

$$[u_k, u_{k+1}, \dots, u_{k+m}; f] \in \mathcal{L}(X^m, Y)$$

*which verifies:*

$$(a') \quad [u_k, u_{k+1}, \dots, u_{k+m}; f](u_{k+m} - u_k) = [u_{k+1}, u_{k+2}, \dots, u_{k+m}; f] \\ - [u_k, u_{k+1}, \dots, u_{k+m-1}; f]$$

- b') *if  $f$  is  $m$  time Fréchet differentiable at  $u_k$ , then*

$$[u_k, u_k, \dots, u_k; f] = \frac{1}{m!} f^{(m)}(u_k).$$

## 2. CONSIDERATIONS ON THE SECANT METHOD

For the approximation of the solution of the operator equation

$$(2.1) \quad f(x) = 0$$

consider the iteration

$$(2.2) \quad x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad n = 1, 2, \dots, x_0, x_1 \in X,$$

It is known that if  $f$  satisfies certain conditions, then the sequence  $(x_n)_{n \geq 0}$  given by (2.2) is well defined (there exists  $[x_{n-1}, x_n; f]^{-1}$  for  $n = 1, 2, \dots$ ) and converges to a solution  $x^*$  of equation (2.1) (see for example [2], [5], [8], [10]).

In the following we shall give some specifications concerning the results obtained in [2]. Then we shall try to obtain conditions that ensure the convergence of the sequence  $(x_n)_{n \geq 0}$  to a solution of (2.1), and, moreover, we shall determine a subset  $E \subset X$  that contains this solution.

In paper [2], where the results obtained in [3] are generalized, the convergence of process (2.2) is studied under the assumptions that  $f$  is Fréchet differentiable on a set  $D \subset X$  and the Fréchet derivative  $f'(\cdot)$  satisfies a Hölder type condition on  $D$ : there exist  $c \in \mathbb{R}$ ,  $c > 0$  and  $p \in (0, 1]$  such that:

$$(2.3) \quad \|f'(x) - f'(y)\| \leq c \|x - y\|^p, \quad \text{for every } x, y \in D.$$

Let  $H_D(c, p)$  denote the set of all applications  $f'$  for which (2.3) holds.

In [2], in addition to the conditions from Definition 1.1, it is assumed that the divided differences of the first order of  $f$  satisfy a Hölder type condition, namely there exist  $l_1, l_2, l_3 \geq 0$ ,  $p \in (0, 1)$  such that for every  $x, y, z \in D$ , the inequality:

$$(2.4) \quad \|[x, y; f] - [y, z; f]\| \leq l_1 \|x - z\|^p + l_2 \|x - y\|^p + l_3 \|y - z\|^p$$

holds.

This condition is useful when divided differences of the second order of  $f$  are unbounded on  $D$ .

Let  $l'_2 = \max\{l_2, l_3\}$ . If  $x^*$  is a simple zero of equation (2.1), then the application  $f'(x^*) \in \mathcal{L}(X, Y)$  has a bounded inverse.

From (2.4) and from the existence and boundness of  $[f'(x^*)]^{-1}$  there exists  $\varepsilon > 0$  such that  $[x, y; f]$  has a bounded inverse for every  $x, y \in \bar{U}(x^*, \varepsilon)$ , where  $\bar{U}(x^*, \varepsilon) = \{x \in X \mid \|x - x^*\| \leq \varepsilon\}$ , namely the application  $B(x, y) = [x, y; f]^{-1}$  is uniformly bounded on  $\bar{U}(x^*, \varepsilon)$ .

In [2] the following theorem was proved:

**THEOREM 2.1.** *Let  $D \subset X$  be an open set and  $f : X \rightarrow Y$ . If:*

- i)  $x^* \in D$  is a simple solution of equation (2.1).
- ii) there exists  $\varepsilon > 0$  and  $b > 0$  such that:

$$\|[x, y; f]^{-1}\| \leq b, \quad \text{for every } x, y \in \bar{U}(x^*, \varepsilon);$$

iii) *there exists a convex set  $D_0$  and a real number  $\varepsilon_1$ ,  $0 < \varepsilon_1 < \varepsilon$ , such that:*

$$f'(\cdot) \in H_{r_0}(c, p), \quad \text{for every } x \in D_0 \text{ and } U(x^*, \varepsilon_1) \subset D_0;$$

iv)  $x_0, x_1 \in \bar{U}(x^*, r)$ , where  $0 < r < \min\{\varepsilon_1, [q(p)]^{-\frac{1}{p}}\}$   
and

$$(2.5) \quad q(p) = \frac{b}{1-p} [2^p (l_1 + l'_2) (1+p) + c],$$

*then the sequence given by (2.2) is well defined, and its elements belong to  $\bar{U}(x^*, r)$ . The sequence converges to the unique solution  $x^*$  of (2.1). Moreover, the following estimation holds*

$$(2.6) \quad \|x_{n+1} - x^*\| \leq \gamma_1 \|x_{n-1} - x^*\|^p \cdot \|x_n - x^*\| + \gamma_2 \|x_n - x^*\|^{1+p}$$

*for  $n$  large enough where  $\gamma_1$  and  $\gamma_2$  are given by*

$$(2.7) \quad \gamma_1 = b (l_1 + l'_2) 2^p$$

$$(2.8) \quad \gamma_2 = \frac{bc}{1+p}$$

The proof is based on the following two lemmas [2]

LEMMA 2.1. *Let  $f : X \rightarrow Y$  and  $D \subset X$  be an open set. If  $f$  is Fréchet differentiable on  $D$  and there exists a convex set  $D_0 \subset D$  such that  $f'(\cdot) \in H_{D_0}(c, p)$ , then for any  $x, y \in D_0$  the following inequality holds:*

$$(2.9) \quad \|f(x) - f(y) - f'(x)(x - y)\| \leq \frac{c}{1+p} \|x - y\|^{p+1}.$$

LEMMA 2.2. *If there exists divided differences  $[x, y; f]$  and inequality (2.4) is verified for all  $x, y, z \in D_0$ , then the equality b) from Definition 1.1 holds for every  $x \in D_0$  and the derivative  $f'$  of  $f$  verifies the relation  $f'(\cdot) \in H_{D_0}[2(l_1 + l_2), p]$ .*

In the proof of Theorem 2.1 the following inequality is obtained first

$$(2.10) \quad \|x_{n+1} - x^*\| \leq [M(r)]^{n+1} \|x_0 - x^*\|, \quad \text{where } 0 < M(r) < 1.$$

from which it follows that the sequence  $(x_n)_{n \geq 0}$  is convergent. In the following, by use of inequality (2.6) obtained in [2], we shall prove that the order of convergence of the sequence given by (2.2) is  $t_1 = \frac{1+(1+4p)^{1/2}}{2}$ , i.e. it is the positive root of the equation:

$$(2.11) \quad t^2 - t - p = 0.$$

For this, besides the hypothesis of Theorem 2.1 we shall suppose that  $x_0$  and  $x_1$  verify

$$\begin{aligned} \text{a')} \quad & \|x^* - x_0\| \leq \alpha d_0; \\ \text{b')} \quad & \|x^* - x_1\| \leq \min\{\alpha d_0^{t_1}, \|x^* - x_0\|\}, \\ & \text{where } 0 < d_0 < 1 \text{ and } \alpha = [q(p)]^{-\frac{1}{p}}. \end{aligned}$$

Using Lemmas 2.1 and 2.2 and hypotheses of Theorem 2.1, from (2.2) we obtain

$$(2.12) \quad \|x_2 - x^*\| \leq \gamma_1 \|x_0 - x^*\|^p \|x_1 - x^*\| + \gamma_2 \|x_1 - x^*\|^{p+1}$$

from which, taking into account a'), b') it follows

$$\|x_2 - x^*\| \leq \alpha d_0^{t_1^2} (\gamma_1 + \gamma_2 d_0^{p(t_1-1)}) \alpha^p,$$

where

$$(\gamma_1 + \gamma_2 d_0^{p(t_1-1)}) \alpha^p = \frac{\gamma_1 + \gamma_2 d_0^{p(t_1-1)}}{\gamma_1 + \gamma_2} < 1$$

that is,

$$(2.13) \quad \|x_2 - x^*\| \leq d_0^{t_1^2}$$

Relations (2.12) and (2.13) imply  $\|x_2 - x^*\| < \|x_1 - x^*\|$ .

Suppose that there exists  $n \in \mathbb{N}$ ,  $n \geq 2$ , such that

$$(a'') \quad \|x_{n-1} - x^*\| \leq \alpha d_0^{t_1^{n-1}}$$

$$(b'') \quad \|x_n - x^*\| \leq \min \{ \alpha d_0^{t_1^n}, \|x_{n-1} - x^*\| \}.$$

If we repeat the above reasoning and take into account (a'') and (b'') we obtain

$$\|x_{n+1} - x^*\| \leq \alpha^{1+p} \cdot d_0^{t_1^{n+1}} (\gamma_1 + \gamma_2 d_0^{p t_1^{n-1}}) \leq \alpha d_0^{t_1^{n+1}},$$

because

$$\alpha^p (\gamma_1 + \gamma_2 d_0^{p t_1^{n-1}}) < 1.$$

Moreover, it can be easily seen that

$$\|x_{n+1} - x^*\| < \|x_n - x^*\|.$$

So far, we have proved the following theorem.

**THEOREM 2.2.** *Under the hypotheses of Theorem 2.1 and if  $x_0$  and  $x_1$  verify a') and b'), where  $\alpha = (q(p))^{-\frac{1}{p}}$  and  $0 < d_0 < 1$ , then for every  $n \in \mathbb{N}$ ,  $x_n \in U(x^*, \alpha)$  and*

$$(2.13') \quad \|x_{n+1} - x^*\| \leq d_0^{t_1^{n+1}}, \quad n = 0, 1, \dots$$

One must notice that inequality (2.13') gives a sharper error bound than (2.10).

In the following we shall establish a result which ensures not only the convergence of the sequences  $(x_n)_{n \geq 0}$  but also the existence of the solution of equation (2.1) in a determined subset of  $X$ .

In this respect, we observe that if  $[x_{n-1}, x_n; f]^{-1}$  exists for every  $n = 1, 2, \dots$  then:

$$(2.14) \quad x_n - [x_{n-1}, x_n; f]^{-1} f(x_n) = x_{n-1} - [x_{n-1}, x_n; f]^{-1} f(x_{n-1})$$

and

(2.15)

$$f(x_{n+1}) = f(x_n) + [x_{n-1}, x_n; f](x_{n+1} - x_n) + ([x_n, x_{n+1}; f] - [x_{n-1}, x_n; f])(x_{n+1} - x_n), \quad n = 1, 2, \dots$$

hold.

Let  $\alpha, B, d_0 \in \mathbb{R}$ ,  $\alpha > 0$ ,  $B > 0$ ,  $d_0 \in (0, 1)$  and

$$S = \left\{ x \in X : \|x - x_0\| \leq \frac{B\alpha d_0}{1 - d_0^{t_1 - 1}} \right\},$$

where  $t_1$  is the positive root of equation (2.11).

**THEOREM 2.3.** *If the divided differences of the first order of the application  $f$  verify condition (2.4) for every  $x, y, z \in S$  and*

- i) *for every  $x, y \in S$ ,  $[x, y; f]^{-1}$  exists and  $\|[x, y, f]^{-1}\| \leq B$*
- ii) *the initial data  $x_0, x_1 \in X$  and  $f$  verify the inequalities*

$$(2.16) \quad \|x_1 - x_0\| < B\alpha d_0, \quad \|f(x_0)\| \leq \alpha d_0 \quad \text{and} \quad \|f(x_1)\| \leq \alpha d_0^{t_1}$$

where

$$(2.17) \quad \alpha = \frac{1}{B^{\frac{1+p}{p}} (l_1 + l_2 + l_3)^{\frac{1}{p}}}$$

the equation (2.1) has at least one solution  $x^* \in S$  which is the limit of the sequence  $(x_n)_{n \geq 0}$  given by (2.2) the order of the convergence of this sequence and the error bound are given by

$$(2.18) \quad \|x^* - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1 - d_0^{t_1^n(t_1 - 1)}}, \quad n = 1, 2, \dots$$

*Proof.* From (2.2), for  $n = 2$  we have

$$\|x_2 - x_1\| \leq B \|f(x_1)\| \leq B\alpha d_0^{t_1}$$

This inequality, together with the first inequality from (2.16), implies.

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq B\alpha d_0(1 + d_0^{t_1 - 1}) \leq \frac{B\alpha d_0}{1 - d_0^{t_1 - 1}},$$

and so  $x_2 \in S$ .

Because  $x_2 \in S$ , from (2.14), (2.15) and using (2.4) we obtain:

$$(2.19) \quad \|f(x_2)\| \leq B^{p+1} \alpha^{p+1} (l_1 + l_2 + l_3 d_0^{t_1^2 - p}) d_0^{t_1^2 - p} \leq \alpha d_0^{t_1^2},$$

since

$$\alpha^p B^{p+1} (l_1 + l_2 + l_3 d_0^{p(t_1 - 1)}) \leq \alpha^p B^{p+1} (l_1 + l_2 + l_3) \leq 1.$$

Suppose

$$(2.20) \quad x_i \in S;$$

$$(2.21) \quad \|f(x_i)\| \leq \alpha d_0^{t_1^i}, \quad \text{hold for } i = \overline{1, k}.$$

Then

$$\begin{aligned} \|x_{n+1} - x_k\| &\leq B\alpha d_0(1 + d_0^{t_1 - 1} + d_0^{t_1^2 - 1} + \dots + d_0^{t_1^k - 1}) \\ &\leq B\alpha d_0(1 + d_0^{(t_1 - 1)} + d_0^{2(t_1 - 1)} + \dots + d_0^{k(t_1 - 1)}) \leq \frac{B\alpha d_0}{1 - d_0^{t_1 - 1}}, \end{aligned}$$

that is,  $x_{k+1} \in S$ .

By the use of the same reasoning as for (2.19) we obtain

$$(2.22) \quad \|f(x_{k+1})\| \leq B^{p+1} \alpha^{p+1} \left( l_1 + l_2 + l_3^{pt_1^{k-1}(t_1-1)} \right) d_0^{t_1^{k-1}(t_1+p)} \leq \alpha_0 d_0^{t_1^{k+1}}.$$

The above relations imply that (2.20) and (2.21) hold for every  $k \in \mathbb{N}$ .

Now notice that  $(x_n)_{n \geq 0}$  is a Cauchy sequence, because

$$(2.23) \quad \|x_{n+s} - x_n\| \leq \sum_{k=n}^{n+s-1} \|x_{k+1} - x_k\| \leq B\alpha \sum_{k=n}^{n+s-1} d_0^{t_1^k} \leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n(t_1-1)}},$$

for any  $n, s \in \mathbb{N}$ ,  $t_1 > 1$  and  $d_0 \in (0, 1)$ .

Let  $x^* = \lim_{n \rightarrow \infty} x_n$ . Then, taking  $s \rightarrow \infty$  in (2.23) we obtain

$$(2.24) \quad \|x^* - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n(t_1-1)}},$$

$n = 0, 1, \dots$ , that is, the inequality (2.18).

For  $n = 0$  we obtain  $x^* \in S$ .

If  $k \rightarrow \infty$  in (2.22) then  $\|f(x^*)\| = 0$ , that is  $x^*$ , is the solution of equation (2.1).  $\square$

### 3. CONSIDERATIONS ON STEFFENSEN METHOD

It is well known that the order of convergence of the secant method can be improved if the elements  $x_{n-1}$  and  $x_n$  from (2.2) are related by an application  $g : X \rightarrow X$ , described in the following.

Consider the sequence  $(x_n)_{n \geq 0}$  generated by

$$(3.1) \quad x_{n+1} - x_n - [x_n, g(x_n); f]^{-1} f(x_n), \quad x_0 \in X,$$

where  $g$  is an operator whose fixed points coincide with solutions of equations (2.1).

Consider  $x_0 \in X$ , the nonnegative real numbers  $B, \varepsilon_0, \rho_0, \alpha, \beta$  and  $q \geq 1$ , where

$$(3.2) \quad \rho_0 = B\alpha (l_1 B^p + l_2 \beta^p + l_3 B^p \cdot \alpha^p) \|f(x_0)\|^{p(q-1)}$$

$$(3.3) \quad \varepsilon_0 = \rho_0 \frac{1}{(p+q-1)} \|f(x_0)\|,$$

the numbers  $l_1, l_2, l_3$  being given by condition (2.4).

$$(3.4) \quad S = \left\{ x \in X : \|x - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}} (1-\varepsilon_0^{p+q-1})} \right\},$$

where  $r = \max\{B, \beta\}$

Concerning the convergence of method (3.1), the following theorem holds:

**THEOREM 3.1.** *If the real numbers  $B, \varepsilon_0, \rho_0, p_0, p, \alpha, \beta, q, l_1, l_2, l_3$  the applications  $f$  and  $g$ , and the element  $x_0$  satisfy the conditions:*

- (i) *for every  $x, y \in S$  there exist  $[x, y; f]^{-1}$  and  $\|[x, y; f]^{-1}\| \leq B$ ;*
- (ii) *for every  $x \in S, \|f(g(x))\| \leq \alpha \|f(x)\|^q$ ;*
- (iii) *for every  $x \in S, \|x - g(x)\| \leq \beta \|f(x)\|$ ;*
- (iv) *the divided differences of the first order of the applications  $f$  verify condition (2.4) for every  $x, y, z \in S$ ;*
- (v)  $\varepsilon_0 < 1$ ,

*then the sequence  $(x_n), n \geq 0$  given by (3.1) is convergent and if  $x^* = \lim x_n$ , then  $f(x^*) = 0$ . Moreover, we have*

$$(3.5) \quad \|x^* - x_n\| \frac{r\varepsilon_0^{(p+q)^n}}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})}.$$

*Proof.* Let  $x_0 \in X$  be such that  $\varepsilon_0$  verifies condition (v). Using similar relations to (2.14) and (2.15) and condition (2.4), we obtain from (3.1)

$$\|x_1 - x_0\| \leq B \cdot \|f(x_0)\| \leq \frac{B\rho_0^{\frac{1}{p+q-1}}}{\rho_0^{\frac{1}{p+q-1}}} \|f(x_0)\| \leq \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})},$$

which means that  $x_1 \in S$ .

In the above inequality we have admitted the relation  $g(x_0) \in S$ , which is implied by (iii).

From (2.14), (2.15), (3.1), (i), (ii) and (iii) we obtain:

$$\begin{aligned} \|f(x_1)\| &\leq \|[g(x_0), x_1; f] - [x_0, g(x_0); f]\| \|x_1 - g(x_0)\| \\ &\leq B\alpha \left( l_1 B^p + l_2 \beta^p + l_3 B^p \alpha^p \|f(x_0)\|^{p(q-1)} \right) \|f(x_0)\|^{p+q} \\ &= \rho_0 \|f(x_0)\|^{p+q}. \end{aligned}$$

From the above inequality there follows

$$\rho_0^{\frac{1}{p+q-1}} \|f(x_1)\| \leq \left( \rho_0^{\frac{1}{p+q-1}} \|f(x_0)\| \right)^{p+q}$$

and if  $\varepsilon_1 = \rho_0^{\frac{1}{p+q-1}} \|f(x_1)\|$  then,

$$\varepsilon_1 \leq \varepsilon_0^{p+q}.$$

It can be easily seen that  $\|f(x_1)\| \leq \|f(x_0)\|$  and  $\rho_1 \leq \rho_0$ , where  $\rho_1 = B\alpha \left[ l_1 B^p + l_2 \beta^p + l_3 B^p \alpha^p \|f(x_1)\|^{p(q-1)} \right]$ . Suppose now that, for  $s = \overline{1, k}$ , the following relations hold  $x_s \in S, \|f(x_s)\| \leq \|f(x_{s-1})\|, \varepsilon_s < \varepsilon_0^{(p+q)^s}$ , where  $\varepsilon_s = \rho_0^{\frac{1}{p+q-1}} \|f(x_s)\|$ .

Using these assumptions and proceeding as above we get

$$(3.6) \quad \|x_{k+1} - x_k\| \leq B \|f(x_k)\| \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{\frac{1}{p+q-1}}}$$

$$(3.7) \quad \|x_{n+1} - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})},$$

showing that  $x_{k+1} \in S$ .

It is also easy to see that

$$(3.8) \quad \|g(x_k) - x_k\| \leq \frac{r\varepsilon_0^{(p+q)^k}}{\rho_0^{\frac{1}{p+q-1}}}$$

whence

$$(3.9) \quad \|g(x_k) - x_0\| \leq \frac{r\varepsilon_0}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})},$$

that is,  $g(x_k) \in S$ .

We obtain further

$$(3.10) \quad \|f(x_{k+1})\| \leq \rho_0 \|f(x_k)\|^{p+q},$$

whence

$$(3.11) \quad \varepsilon_{k+1} \leq \varepsilon_0^{(p+q)^{k+1}}, \text{ when } \varepsilon_{k+1} = \rho_0^{\frac{1}{p+q-1}} \|f(x_{k+1})\|.$$

From (3.6) it follows that, for every  $s, n \in \mathbb{N}$ ,

$$(3.12) \quad \|x_{n+s} - x_n\| \leq \frac{B\varepsilon_0^{(p+q)^n}}{\rho_0^{\frac{1}{p+q-1}}(1-\varepsilon_0^{p+q-1})}$$

and by (v) the sequence  $(x_n)_{n \geq 0}$  is fundamental, hence convergent. If  $x^* = \lim_{n \rightarrow \infty} x_n$ , from (3.12), for  $s \rightarrow \infty$ , we get (3.5) and from (3.11) it follows that  $x^*$  is a solution of (2.1).

From (3.5), for  $n = 0$  we have that  $x^* \in S$ .  $\square$

#### 4. CONSIDERATIONS CONCERNING NEWTON'S METHOD

Consider the sequence given by Newton's method,

$$(4.1) \quad x_{n+1} = x_n - [f'(x_n)]^{-1} f(x_n), \quad n = 0, 1, \dots, x_0 \in X$$

let  $S(x_0, r) = \{x \in X \mid \|x - x_0\| \leq r\}$ , where  $r \in \mathbb{R}$ ,  $r > 0$ .

Concerning the convergence of this sequence we have the following theorem.

**THEOREM 4.1.** *If the application  $f$  is Fréchet differentiable on  $S(x_0, r)$ , the Fréchet derivative  $f'$  satisfies (2.3) for every  $x, y \in S(x_0, r)$  and the following conditions hold:*

- (i)  $[f'(x_0)]^{-1}$  exists and  $\|[f'(x_0)]^{-1}\| \leq d$ ;
- (ii)  $cr^p d < 1$ ;
- (iii)  $\rho_0 = \alpha^{\frac{1}{p}} \|f(x_0)\| < 1$ , where  $\alpha = \frac{c\beta^{1+p}}{1+p}$  and  $\beta = \frac{d}{1-cdr^p}$ ,
- (iv)

$$\frac{\beta \|f(x_0)\|}{1-\alpha \|f(x_0)\|^p} \leq r,$$

then,



- (j)  $x_n \in S(x_0, r)$ , for every  $n \in \mathbb{N}$ ,
- (jj) there exists  $\Gamma_n = [f'(x_n)]^{-1}$  for every  $n \in \mathbb{N}$  and  $\|\Gamma_n\| < \frac{d}{1-dcr^p}$ ,
- (jjj)  $\|x_{n+1} - x_n\| \leq \beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n}$ ;
- (jv) the sequence  $(x_n)_{n \geq 0}$  is convergent, and if  $x^* = \lim_{n \rightarrow \infty} x_n$  then  $f(x^*) = 0$  and

$$(4.2) \quad \|x^* - x_n\| \leq \frac{\beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n}}{1 - \rho_0^{(1+p)^n}}, \quad n \in \mathbb{N}.$$

*Proof.* By (4.1), for  $n = 1$  we get  $x_1 = x_0 - [f'(x_0)]^{-1} f(x_0)$ , and from (i)  $\|x_1 - x_0\| \leq d \|f(x_0)\| \leq \beta \|f(x_0)\| \leq r$ , that is,  $x_1 \in S(x_0, r)$ .  
By (2.3) and ii) it follows

$$\|[f'(x_0)]^{-1} [f'(x_0) - f'(x_1)]\| \leq dc \|x_1 - x_0\|^p \leq dcr^p < 1,$$

whence  $[f'(x_1)]^{-1}$  exists and

$$\|[f'(x_1)]^{-1}\| \leq \frac{d}{1-dcr^p} = \beta.$$

From (2.3) it follows

$$\begin{aligned} \|f(x_1)\| &= \|f(x_1) - f(x_0) - f'(x_0)(x_1 - x_0)\| \\ &\leq \frac{c}{p+1} \|x_1 - x_0\|^{p+1} \leq \frac{c\beta^{p+1}}{p+1} \|f(x_0)\|^{p+1} \end{aligned}$$

and if

$$\rho_1 = \alpha^{\frac{1}{p}} \|f(x_1)\|$$

then

$$\rho_1 \leq \rho_0^{1+p}, \quad \|x_2 - x_1\| \leq \beta \alpha^{-\frac{1}{p}} \rho_0^{1+p}.$$

If  $\rho_i = \alpha^{\frac{1}{p}} \|f(x_i)\|$  and

- (a)  $x_i \in S(x_0, r)$ ,  $i = \overline{1, k}$ ,
- (b)  $\|x_{i+1} - x_i\| \leq \beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^i}$ ,  $i = \overline{1, k-1}$ ,
- (c)  $\rho_i \leq \rho_0^{(1+p)^i}$ ,  $i = \overline{1, k}$ ,

then we get by (a), (2.3) and (i)

$$\|[f'(x_0)]^{-1} [f'(x_k) - f'(x_0)]\| \leq dc \|x_k - x_0\|^p \leq cdr^p < 1.$$

It follows that

$$\|[f'(x_k)]^{-1}\| \leq \frac{d}{1-dcr^p} = \beta.$$

From (4.1) and from the above inequality we get that:

$$\|x_{k+1} - x_k\| \leq \beta \|f(x_k)\| \leq \beta \alpha^{-\frac{1}{p}} \rho_k \leq \beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^k},$$

which by (b) implies

$$\|x_{k+1} - x_0\| \leq \frac{\beta \|f(x_0)\|}{1 - \alpha \|f(x_0)\|^p} \leq r,$$

that is,  $x_{k+1} \in S(x_0, r)$ .

Using the assumptions of the theorem we get that

$$\begin{aligned} \|f(x_{k+1})\| &= \|f(x_{k+1}) - f(x_k) - f'(x_k)(x_{k+1} - x_k)\| \\ &\leq \frac{c}{p+1} \|x_{k+1} - x_k\|^{p+1} \leq \alpha^{-\frac{1}{p}} \rho_k^{p+1}, \end{aligned}$$

whence

$$\rho_{k+1} \leq \rho_0^{(1+p)^{k+1}}.$$

For every  $m, n \in \mathbb{N}$

$$\|x_{m+n} - x_n\| \leq \frac{\beta \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n}}{1 - \rho_0^{p(1+p)^n}}$$

which, together with  $\rho_0 < 1$ , show that  $(x_n)_{n \geq 0}$  is a Cauchy sequence. If  $x^* = \lim_{n \rightarrow \infty} x_n$  then for  $m \rightarrow \infty$  in the above inequality, we get (4.2), and from

$$\|f(x_n)\| \leq \alpha^{-\frac{1}{p}} \rho_0^{(1+p)^n},$$

for  $n \rightarrow \infty$ , we get  $f(x^*) = 0$ . □

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