APPROXIMATION OF THE ROOTS OF EQUATIONS BY AITKEN-STEFFENSEN-TYPE MONOTONIC SEQUENCES(1)

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Abstract. The aim of this paper is to study the conditions under which the well-known Aitken-Steffensen method for solving equations leads to monotonic sequences whose terms approximate (from the left and from the right) the root of an equation. The convergence order and efficiency index of this method are also studied in the general case and then in various particular cases.

1. Introduction

From a practical standpoint, in order to approximate the roots of equations it is advantageous to use methods which lead to monotonic sequences. In this paper we shall use a single iterative process to determine an increasing sequence $(x_n)_{n\geq 0}$ and a decreasing one $(y_n)_{n\geq 0}$, both converging to a root $\bar{x}$ of the equation $f(x) = 0$. Such a procedure has the advantage of allowing, at each iteration step, an approximation error checking, i.e.

$$\max\{\bar{x} - x_n, y_n - \bar{x}\} \leq y_n - x_n.$$

As it is well known, such sequences can be generated by applying simultaneously two methods, e.g. Newton's method and the chord method. In the sequel we shall show that, in conditions similar to those imposed to the above mentioned methods, the Aitken-Steffensen method generates two sequences which fulfil the above inequality.

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In Section 2 we give sufficient conditions for the general method of Aitken-Steffensen ((1.3) below) to generate two monotonic sequences, both converging to the solution $\bar{x}$ of equation (1.1) below. In this way some similar results obtained in [4] are completed, and the proofs given in [5] are made simpler to a certain extent.

In Section 3 we study some particular cases of the method (1.3), namely the methods (3.1) and (3.3) below. In Section 4 there is indicated a way to construct the auxiliary functions $g_1, g_2,$ or g, required by (1.3) or (3.1), in relatively simple conditions as the monotonicity and convexity of the function f. In Section 5 the convergence orders and the efficiency indices of the methods (1.3) and (3.1) are studied, concluding that the method (3.1) has a higher efficiency index. This one is the same as that of Newton's method, but is given conditions, the method (3.1) as well as (1.3) and (3.3), provide in addition bilateral approximations for the root of the equation (1.1).

So, let $I = [a, b]$, $a < b$, be an interval of the real axis, and consider the equation

$$f(x) = 0 \tag{1.1}$$

where $f : I \to \mathbb{R}$. Besides (1.1), consider two more equations

$$\begin{aligned} x - g_1(x) &= 0, \\ x - g_2(x) &= 0, \end{aligned} \tag{1.2}$$

where $g_1, g_2 : I \to \mathbb{R}$.

The Aitken-Steffensen method consists in constructing the sequences $(g_2(g_1(x_n)))_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ by means of the following iterative process

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n); g_2(g_1(x_n)); f]}, \qquad n = 0, 1, \ldots, x_0 \in I, \tag{1.3}$$

where $[u, v; f]$ stands for the first order divided difference of f on the points $u, v \in I$. We shall denote by $[u, v, w; f]$ the second order divided difference of f on $u, v, w \in I$.

As it is well known [6, pp. 268–269], and will also be shown in Section 5, the convergence order of the sequence $(x_n)_{n\geq 0}$ generated by (1.3) is at least 2, but it generally depends on the convergence orders of the sequences $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ generated by $y_{n+1} = g_1(y_n)$, $y_0 \in I$; $z_{n+1} = g_2(z_n)$, $z_0 \in I$; $n = 0, 1, \ldots,$;[2], [5].

Another advantage of the iteration method (1.3) is the following: if equation (1.1) is given, then (as we shall see in Section 4) we have at our disposal many possibilities to construct the functions g_1 and g_2 from equations (1.2). On the other hand, hypotheses concerning the differentiability of f on the whole interval I are not needed.

If $g : I \to \mathbb{R}$ is a function, we shall adopt the following definitions concerning the monotonicity and convexity of g:

Definition 1.1. *The function $g : I \to \mathbb{R}$ is said to be increasing (nondecreasing; decreasing; nonincreasing) on I if for every $x, y \in I$ the relations $[x, y; f] > 0 (\geq 0; < 0; \leq 0)$ hold respectively.*

Definition 1.2. *The function $g : I \to \mathbb{R}$ is convex (nonconcave; concave; nonconvex) on I if for every $x, y, z \in I$ the relations $[x, y, z; f] > 0 (\geq 0; < 0; \leq 0)$ hold respectively.*

2. Convergence of the Aitken-Steffensen method

We shall adopt the following hypotheses concerning the functions f, g_1 and g_2;

(a) f, g_1, g_2 are continuous on I;
(b) g_1 is increasing on I;
(c) g_2 is decreasing on I;
(d) equations (1.1) and (1.2) have only one common root $\bar{x} \in]a, b[$;
(e) for every $x, y \in I$, the relation $0 < [x, y; g_1] \leq 1$ holds.

Concerning the problem stated in Section 1, the following theorem holds:

Theorem 2.1. *Suppose that f, g_1 and g_2 fulfil conditions* (a)–(e) *and, in addition,*

(i_1) *f is increasing and convex on I, and there exists $f'(\bar{x})$;*
(ii_1) *there exists $x_0 \in I$ such that $f(x_0) < 0$ and $g_2(g_1(x_0)) \in I$.*

Then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$, $(g_2(g_1(x_n)))_{n\geq 0}$ generated by (1.3), *where x_0 fulfils condition* (ii_1), *have the following properties:*

(j_1) *the sequences $(x_n)_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ are increasing and bounded;*
(jj_1) *the sequence $(g_2(g_1(x_n)))_{n\geq 0}$ is decreasing and bounded;*
(jjj_1) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$;
(jv_1) $x_n \leq g_1(x_n) \leq x_{n+1} \leq \bar{x} \leq g_2(g_1(x_n))$, $n = 0, 1, \ldots,$
and $\max\{\bar{x} - x_{n+1}, g_2(g_1(x_n)) - \bar{x}\} \leq g_2(g_1(x_n)) - x_{n+1}$.

Proof. Since f is increasing on I and $f(x_0) < 0$, it follows that $x_0 < \bar{x}$. For $x < y$, by (e) we get $g_1(y) - g_1(x) \leq y - x$, which, for $y = \bar{x}$, leads to $x - g_1(x) \leq 0$ if $x < \bar{x}$ and, analogously, $x - g_1(x) \geq 0$ if $x > \bar{x}$. By b) and $x_0 < \bar{x}$ we obtain $g_1(x_0) < g_1(\bar{x})$, i.e. $g_1(x_0) < \bar{x}$, while from the above inequalities it follows $x_0 \leq g_1(x_0)$. By (c) and $g_1(x_0) < \bar{x}$ we get $g_2(g_1(x_0)) > g_2(\bar{x})$, that is, $g_2(g_1(x_0)) > \bar{x}$. By (i_1) and $g_1(x_0) < \bar{x}$ we obtain $f(g_1(x_0)) < 0$ which, together with $[g_1(x_0), g_2(g_1(x_0)); f] > 0$ and (1.3), shows that $x_1 > g_1(x_0)$.

It is easy to see that the following identities

$$g_1(x) - \tfrac{f(g_1(x))}{[g_1(x), g_2(g_1(x)); f]} = g_2(g_1(x)) - \tfrac{f(g_2(g_1(x)))}{[g_1(x), g_2(g_1(x)); f]} \tag{2.1}$$

$$f(z) = f(x) + [x, y; f](z - x) + [x, y, z; f](z - x)(z - y,) \tag{2.2}$$

hold for every $x, y, z \in I$.

Since $g_2(g_1(x_0)) > \bar{x}$, it follows that $f(g_2(g_1(x_0))) > 0$ and, using (2.1) and (1.3), we get $x_1 < g_2(g_1(x_0))$. If we put in (2.2) $z = x_1$, $x = g_1(x_0)$, $y = g_2(g_1(x_0))$, and take into account (1.3), we get

$$f(x_1) = [g_1(x_0), g_2(g_1(x_0)), x_1; f](x_1 - g_1(x_0))(x_1 - g_2(g_1(x_0))),$$

from which, considering the convexity of f and the inequalities $x_1 - g_1(x_0) > 0$ and $x_1 - g_2(g_1(x_0)) < 0$, one obtains $f(x_1) < 0$, that is, $x_1 < \bar{x}$.

In this way we have obtained

$$x_0 \leq g_1(x_0) < x_1 < \bar{x} < g_2(g_1(x_0)). \tag{2.2'}$$

In order that the above reasoning may be repeated we still have to show that x_1 verifies the last condition of (ii_1), namely $g_2(g_1(x_1)) \in I$. Indeed, from $x_0 < x_1$ and (b) it follows $g_1(x_0) < g_1(x_1)$, which, by (c), leads to $g_2(g_1(x_0)) > g_2(g_1(x_1))$. It remains to show that $g_2(g_1(x_0)) > \bar{x}$. Suppose that $g_2(g_1(x_0)) \leq \bar{x}$, that is, $g_2(g_1(x_0)) \leq g_2(\bar{x})$, implying $g_1(x_0) \geq \bar{x}$, which contradicts (2.2′).

Consider $x_n \in I$ for which $f(x_n) < 0$ and $g_2(g_1(x_n)) \in I$, $n \in \mathbb{N}$. Repeating the above reasoning, we obtain the inequalities

$$x_n \leq g_1(x_n) < x_{n+1} < \bar{x} < g_2(g_1(x_n)),\ n = 0, 1, \ldots, \tag{2.3}$$

In this way conclusions (j_1), (jj_1) and (jv_1) of Theorem 2.1 are proved. To prove (jjj_1), denote $l = \lim x_n$, $l_1 = \lim g_1(x_n)$, $l_2 = \lim g_2(g_1(x_n))$. We shall show that $l = l_1 = l_2 = \bar{x}$.

By (2.3) and (a), we deduce:

$$l \le g_1(l) \le l \le \bar{x} \le g_2(g_1(l));$$

but $l_1 = g_1(l)$, therefore $l = l_1$; i.e. $g_1(l) = l$. From this relation and (1.3), using the fact that there exists $f'(\bar{x})$ and f is increasing, we get $f(l) = 0$, that is, $l = \bar{x}$. Since $l_2 = g_2(l) \ge \bar{x}$, it follows $f(g_2(l)) \ge 0$, while form (2.1) it follows that $l \ge g_2(l)$, which, together with $l \le g_2(l)$, leads to $l = g_2(l) = \bar{x}$. □

In can be shown analogously that the following theorems hold, too:

Theorem 2.2. *Suppose that* f, g_1, g_2 *fulfil conditions* (a)–(e), *and in addition*

(i_2) *f is increasing and concave on I, and there exists $f'(\bar{x})$;*
(ii_2) *there exists $x_0 \in I$ for which $f(x_0) > 0$ and $g_2(g_1(x_0)) \in I$.*

Then the sequences $(x_n)_{n\ge0}$, $(g_1(x_n))_{n\ge0}$, $g_2(g_1(x_n))_{n\ge0}$ *generated by* (1.3), *have the following properties:*

(j_2) $(x_n)_{n\le0}$, $(g_1(x_n))_{n\ge0}$ *are decreasing and bounded;*
(jj_2) $(g_2(g_1(x_n)))_{n\ge0}$ *is increasing and bounded;*
(jjj_2) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$, *and the following relations hold*

$$g_2(g_1(x_n)) < \bar{x} < x_{n+1} < g_1(x_n) \le x_n,\ n = 0, 1, \dots,$$

$$\max\{\bar{x} - g_2(g_1(x_n)),\ x_{n+1} - \bar{x}\} \le x_{n+1} - g_2(g_1(x_n)).$$

Theorem 2.3. *Suppose that* f, g_1, g_2 *fulfil conditions* (a)–(e) *and in addition*

(i_3) *f is decreasing and convex on I, and there exists $f'(\bar{x})$;*
(ii_3) *there exists $x_0 \in I$ for which $f(x_0) < 0$ and $g_2(g_1(x_0)) \in I$.*

Then the sequences $(x_n)_{n\ge0}$, $(g_1(x_n))_{n\ge0}$, $(g_2(g_1(x_n)))_{n\ge0}$ *generated by* (1.3) *fulfil the conclusions of Theorem* 2.2.

Theorem 2.4. *Suppose that* f, g_1, g_2 *fulfil conditions* (a)–(e) *and in addition*

(i_4) *f is decreasing and concave on I, and there exists $f'(\bar{x})$;*
(ii_4) *there exists $x_0 \in I$ for which $f(x_0) > 0$ and $g_2(g_1(x_0)) \in I$.*

Then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$, $(g_2(g_1(x_n)))_{n\geq 0}$ *generated by* (1.3), *fulfil the conclusions of Theorem* 2.1.

3. Particular cases

An interesting particular case of the procedure (1.3) is obtained taking $g_1(x) = x$ for every $x \in I$. In this way one obtains Steffensen's method, namely

$$x_{n+1} = x_n - \tfrac{f(x_n)}{[x_n, g(x_n); f]}, \qquad n = 0, 1, \ldots, x_0 \in I, \tag{3.1}$$

where g stands for g_2.

One observes easily that in this case hypotheses (a), (b), (d) and (e) are automatically fulfilled by g_1. For the study of the convergence of the sequence $(x_n)_{n\geq 0}$ generated by (3.1), we have to adopt the following hypotheses:

(a_1) the functions f and g are continuous on I;
(b_1) the function g is decreasing on I;
(c_1) equations (1.1) and $x = g(x)$ have only one common root $\bar{x} \in]a, b[$.

With these specifications, the theorems stated in Section 2 yield:

Corollary 3.1. *Suppose that* f *and* g *fulfil conditions* (a_1)–(c_1) *and in addition* f *is increasing and convex on* I; *there exists* $f'(\bar{x})$, *and the initial point* x_0 *in* (3.1) *can be chosen such that* $f(x_0) < 0$ *and* $g(x_0) \in I$. *Then the sequence* $(x_n)_{n\geq 0}$ *is increasing and bounded, while the sequence* $(g(x_n))_{n\geq 0}$ *is decreasing and bounded. Moreover, the relations* $\bar{x} = \lim x_n = \lim g(x_n)$ *and*

$$\begin{aligned} x_n \leq \bar{x} \leq g(x_n); \\ \max\{\bar{x} - x_n, g(x_n) - \bar{x}\} \leq g(x_n) - x_n \end{aligned}$$

hold.

Corollary 3.2. *Suppose that* f *and* g *fulfil conditions* (a_1)–(c_1) *and in addition* f *is increasing and concave on* I; *there exists* $f'(\bar{x})$, *and* x_0 *from* (3.1) *can be chosen such that* $f(x_0) > 0$ *and* $g(x_0) \in I$. *Then* $(x_n)_{n\geq 0}$ *is decreasing and bounded, while* $(g(x_n))_{n\geq 0}$ *is increasing and bounded. Moreover, the following* *relations:* $\bar{x} = \lim x_n = \lim g(x_n)$ *and*

$$g(x_n) \leq \bar{x} \leq x_n;$$
$$\max\{\bar{x} - g(x_n), x_n - \bar{x}\} \leq x_n - g(x_n)$$

hold.

Corollary 3.3. *Suppose that f and g fulfil conditions* (a$_1$)–(c$_1$) *and in addition f is decreasing and convex on I; there exists $f'(\bar{x})$, and x_0 from* (3.1) *can be chosen such that $f(x_0) < 0$ and $g(x_0) \in I$. Then $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ verify the conclusions of Corollary* 3.2.

Corollary 3.4. *Suppose that f and g fulfil conditions* (a$_1$)–(c$_1$) *and in addition f is decreasing and concave on I, there exists $f'(\bar{x})$ and x_0 from* (3.1) *can be chosen such that $f(x_0) > 0$ and $g(x_0) \in I$. Then $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ verify the conclusions of Corollary* 3.1.

Another interesting particular case is that in which f has the form:

$$f(x) = x - g(x) = 0. \tag{3.2}$$

In this case the iterative method (3.1) will have the form

$$x_{n+1} = x_n - \frac{(x_n - g(x_n))^2}{g(g(x_n)) - 2g(x_n) + x_n}, \qquad n = 0, 1, \ldots, x_0 \in I, \tag{3.3}$$

whose convergence follows easily by Corollaries 3.1–3.4. The following results simplify the conditions imposed to g and x_0 in [1]:

Corollary 3.5. *Suppose that g is continuous, decreasing and convex on I; equation* (3.2) *has a root $\bar{x} \in]a, b[$, there exists $g'(\bar{x})$, and $x_0 \in I$ can be chosen such that $x_0 < g(x_0)$ and $g(x_0) \in I$. Then the sequences $(x_n)_{n\geq 0}$ and $(g(x_n))_{n\geq 0}$ verify the conclusions of Corollary* 3.1.

Corollary 3.6. *Suppose that g is continuous, decreasing and concave on I, equation* (3.2) *has a root $\bar{x} \in]a, b[$, there exists $g'(\bar{x})$, and $x_0 \in I$ can be chosen such* *that* $x_0 > g(x_0)$ *and* $g(x_0) \in I$*. Then the sequences* $(x_n)_{n\geq 0}$ *and* $(g(x_n))_{n\geq 0}$ *verify the conclusions of Corollary* 3.2.

The restriction imposed to f in Corollaries 3.3 and 3.4 (f to be decreasing) can no more be reached in this case, because in those corollaries the condition on g to be decreasing is essential, and is easy to see that if g decreases the f is increasing.

4. Determination of auxiliary functions

In what follows we shall show that, under reasonable hypotheses concerning the monotonicity and convexity of f, we have at our disposal various ways to choose g_1, g_2 and g such that the hypotheses of the theorems and corollaries stated above are verified.

1. Let us admit that $f : [a, b] \to \mathbb{R}$ is differentiable on $[a, b]$, and denote by $f'(a)$ the right-hand derivative in a and by $f\prime(b)$ the left-hand derivative in b; we also admit that f is increasing and convex on $[a, b]$. Suppose that the equation $f(x) = 0$ has a root $\bar{x} \in]a, b[$. Then we choose $g_1(x) = x - f(x)/f'(b)$ and $g_2(x) = x - f(x)/f'(a)$.

Since f is convex, if follows that f' is increasing on $[a, b]$, hence $0 < f'(a) < f'(x) < f'(b)$ for every $x \in]a, b[$. In this case we have $g_1'(x) = 1 - f'(x)/f'(b) > 0$ for every $x \in]a, b[$ and $g_2'(x) = 1 - f'(x)/f'(a) < 0$ for every $x \in]a, b[$. We choose then a subinterval $[\alpha, \beta] \subset]a, b[$ for which $\bar{x} \in [\alpha, \beta]$. If we suppose in addition that there exists $x_0 \in [\alpha, \beta]$ for which $f(x_0) < 0$ and $g_2(g_1(x_0)) \in [\alpha, \beta]$ then the above constructed functions f, g_1 and g_2 verify the hypotheses of Theorem 2.1. It is easy to see that if we choose $g_1(x) = x - \lambda_1 f(x)$ and $g_2(x) = x - \lambda_2 f(x)$, with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 > f'(b)$ and $0 < \lambda_2 < f'(a)$, then g_1 and g_2 also verify the hypotheses of Theorem 2.1.

2. If f is differentiable, decreasing and convex on $[a, b]$, then g_1 and g_2 can be chosen as in case 1. If there exists $x_0 \in [\alpha, \beta]$ such that $f(x_0) < 0$ and $g_2(g_1(x_0)) \in [\alpha, \beta]$, then the hypotheses of Theorem 2.3 are verified.

3. If f is differentiable, decreasing and concave on $[a, b]$, then we may put $g_1(x) = x - f(x)/f'(a)$ and $g_2(x) = x - f(x)/f'(b)$. If there exists $x_0 \in [\alpha, \beta]$ for which $f(x_0) > 0$ and $g_2(g_1(x_0)) \in [\alpha, \beta]$, then f, g_1 and g_2 fulfil the conditions of Theorem 2.4.

4. If f is differentiable, increasing and concave on $[a, b]$, then g_1 and g_2 can be chosen as in case 3. If there exists $x_0 \in [\alpha, \beta]$ such that $f(x_0) > 0$ and $g_2(g_1(x_0)) \in I$, then f, g_1 and g_2 verify the hypotheses of Theorem 2.2.

5. The function g appearing in Section 3, Procedure (3.1) can be chosen in the following ways:

a) If f is differentiable, increasing and convex on $[a,b]$, then we can choose $g(x) = x - f(x)/f'(a)$, and if x_0 verifies $f(x_0) < 0$ and $g(x_0) \in [\alpha, \beta]$, then the hypotheses of Corollary 3.1 are verified; while if f is decreasing and convex and there exists $x_0 \in [\alpha, \beta]$ for which $f(x_0) < 0$ and $g(x_0) \in [\alpha, \beta]$, the hypotheses of Corollary 3.3 are verified.

b) If f is differentiable, decreasing and concave on $[a,b]$, we choose $g(x) = x - f(x)/f'(b)$. If there exists $x_0 \in [\alpha, \beta]$ such that $f(x_0) > 0$ and $g(x_0) \in [\alpha, \beta]$, then the hypotheses of Corollary 3.4 are verified; while if f is increasing and concave and there exists $x_0 \in [a,b]$ such that $f(x_0) > 0$ and $g(x_0) \in [\alpha, \beta]$, the hypotheses of Corollary 3.2 are verified.

5. Convergence order and efficiency index

To fix the ideas, we shall consider hereafter, besides (1.1), another equation, equivalent to (1.1), of the form

$$x - h(x) = 0 \tag{5.1}$$

where $h : I \to \mathbb{R}$. We shall also consider a sequence $(x_n)_{n\geq 0}$, $x_n \in I$, which, together with h and f, verifies the properties:

(a) $h(x_n) \in I$ for every $n \in \mathbb{N}$;
(b) $(x_n)_{n\geq 0}$ and $(h(x_n))_{n\geq 0}$ are convergent and $\lim x_n = \lim h(x_n) = \bar{x}$, where $\bar{x}$ is the root of equation (1.1);
(c) $[x, y; f] \neq 0$ for every $x, y \in I$;
(d) f is differentiable at $x = \bar{x}$.

We shall adopt the following definition of the convergence order of the sequence $(x_n)_{n\geq 0}$:

Definition 5.1. *The sequence $(x_n)_{n\geq 0}$ is said to have the convergence order $p \in \mathbb{R}$ with respect to h if there exists*

$$\alpha = \lim \frac{\ln |h(x_n) - \bar{x}|}{\ln |x_n - \bar{x}|} \tag{5.2}$$

and $\alpha = p$.

If $x_{n+1} = h(x_n)$, $n = 0, 1, \dots,$ *the above definition reduces to the usual one* [2], [3].

The following theorem holds:

Theorem 5.1. *If* h *and* f *and* $(x_n)_{n\geq 0}$ *verify the properties* (a)–(d), *then the necessary and sufficient condition for* $(x_n)_{n\geq 0}$ *to have the convergence order* $p \in \mathbb{R}$, $p > 0$, *with respect to* h *is to exist*

$$\beta = \lim \frac{\ln |f(h(x_n))|}{\ln |f(x_0)|} \tag{5.3}$$

and $\beta = p$.

Proof. Supposing (5.2) or (5.3) to be true and taking into account (a)–(d), it follows

$$\lim \frac{\ln |h(x_n) - \bar{x}|}{\ln |x_n - \bar{x}|} = \lim \frac{\ln |f(h(x_n))| - \ln |[h(x_n), \bar{x}; f]|}{\ln |f(x_n)| - \ln |x_n, \bar{x}; f|}$$
$$= \lim \frac{\ln |f(h(x_n))|}{\ln |f(x_n)|} \cdot \frac{1 - \frac{\ln|[h(x_n),\bar{x};f]|}{\ln|f(h(x_n))|}}{1 - \frac{\ln|[x_n,\bar{x};f]|}{\ln|f(x_n)|}} = \lim \frac{\ln |f(h(x_n))|}{\ln |f(x_n)|}.$$

The equality of these two limits proves the validity of the theorem. □

Consider now the functions g_1, g_2 appearing in equations (1.2) and let the function h be given by

$$h(x) = g_1(x) - \frac{f(g_1(x))}{[g_1(x), g_2(g_1(x)); f]}. \tag{5.4}$$

Concerning the convergence order of the sequence $(x_n)_{n\geq 0}$ generated by (1.3), the following theorem holds:

Theorem 5.2. *Suppose that the functions* f, g_1 *and* g_2 *and the initial point* x_0 *fulfil the conditions of Theorem* 2.1. *If there exists* $f''(\bar{x})$ *and in addition the sequence* $(x_n)_{n\geq 0}$ *has the convergence order* p_1 *with respect to* g_1, *while* $(g_1(x_n))_{n\geq 0}$ *has the convergence order* p_2 *with respect to* g_2, *then the sequence* $(x_n)_{n\geq 0}$ *has* *the convergence order* $p_1(p_2+1)$ *with respect to the function* h *given by* (5.4).

Proof. At first observe that, under the stated hypotheses, we may use (5.3) to determine the convergence order. The same hypotheses also lead to

$$\lim \frac{\ln|f(g_1(x_n))|}{\ln|f(x_n)|} = p_1; \tag{5.5}$$

$$\lim \frac{\ln|f(g_2(g_1(x_n)))|}{\ln|f(g_1(x_n))|} = p_2. \tag{5.6}$$

Using (2.1), (2.2) and the procedure (1.3), we obtain

$$f(x_{n+1}) = [g_1(x_n), g_2(g_1(x_n)), x_{n+1}; f] \frac{f(g_1(x_n)) \cdot f(g_2(g_1(x_n)))}{[g_1(x_n), g_2(g_1(x_n)); f]^2},$$

from which, by (5.5) and (5.6), it follows easily the equality

$$\lim \frac{\ln|f(h(x_n))|}{\ln|f(x_n)|} = p_1(1+p_2).$$

□

Remark 5.1. Theorem 5.2 remains valid if the hypotheses of Theorem 2.1 are replaced by those of Theorems 2.2, 2.3, or 2.4.

An analogous theorem holds in the case of the procedure (1.3), too.

Theorem 5.3. *If* f *and* g *verify the hypotheses of Corollary* 3.1, *if there exists* $f''(\bar{x})$, *and in addition the sequence* $(x_n)_{n\geq 0}$ *generated by* (3.1) *has the convergence order* p *with respect to* g, *then this sequence has the convergence order* $p+1$ *with respect to the function* h *given by*

$$h(x) = x - \frac{f(x)}{[x, g(x); f]}.$$

Remark 5.2. Theorem 5.3 remains valid if the hypotheses of Corollary 3.1 are replaced by those of Corollaries 3.2, 3.3, or 3.4.

Remark 5.3. Returning to the functions g_1 and g_2 determined in Section 4, ($g_1 = x - f(x)/A, g_2 = x - f(x)/B$, where $A = f'(b), B = f'(a)$, or $A = f'(a), B = f'(b)$, according to the case under consideration), it is easy to see that the convergence order of the sequence generated by (1.3) is equal to 2.

In the case of the procedure (3.1), for $g(x) = x - f(x)/A$ or $g(x) = x - f(x)/B$ (according to the considered situation), the convergence order is also equal to 2.

In the sequel we shall deal with the efficiency index of the iterative procedures studied along the previous sections.

According to A. M. Ostrowski's [3] definition for the efficiency index of an iterative procedure, and taking into account Theorems 5.1, 5.2 or 5.3 for the iterative procedures (1.3), (3.1), or (3.3), the efficiency index is expressed as

$$E = p^{1/m}$$

Here p is the convergence order of the sequence generated by one of these procedures, while m stands for the number of values of functions to be computed at each iteration step.

Taking into consideration Remark 5.3, it follows that, choosing g_1, g_2, or g as in Section 4, the efficiency index of the procedure (3.1) is $E = \sqrt[3]{2}$, while that of (3.3) is $E = \sqrt{2}$.

From this standpoint, the procedure (3.1) has the efficiency index equal to that of Newton's method.

In the conditions of Theorem 5.2 for the generalized procedure of Aitken-Steffensen (1.3), the efficiency index is given by $E = \sqrt[4]{p_1(p_2+1)}$. Analogously, in the conditions of Theorem 5.3, the efficiency index of the method (3.1) is $E = \sqrt[3]{p+1}$.

6. Numerical examples

1. Consider the equation

$$f(x) := x - 2\arctan x = 0,$$

for $x \in \left[\frac{3}{2}, 3\right]$. Since f is increasing and convex on $\left[\frac{3}{2}, 3\right]$, we can choose g_1 and g_2 as in the case 1 (Section 4). We obtain

$$g_1(x) = \tfrac{1}{4}(10 \ \arctan \ x - x);$$
$$g_2(x) = \tfrac{1}{5}(26 \ \arctan \ x - 8x).$$

Taking $x_0 = \frac{3}{2}$, the functions f, g_1 and g_2 verify the hypotheses of Theorem 2.1 on the interval $\left[\frac{3}{2}, 3\right]$. Applying the procedure (1.3), we get the following results:

n	x_n	$g_1(x_n)$	$g_2(g_1(x_n))$	$f(x_n)$
0	1.50000000000000	2.08198430811832	2.50854785469606	$-4.6 \cdot 10^{-01}$
1	2.32357265230323	2.33006829103803	2.33195667567199	$-5.1 \cdot 10^{-03}$
2	2.33112222668589	2.33112235050042	2.33112238618252	$-9.9 \cdot 10^{-08}$
3	2.33112237041442	2.33112237041442	2.33112238041442	$-3.5 \cdot 10^{-17}$

2. Consider the equation

$$f(x) := x - \arcsin \frac{x-1}{\sqrt{2(x^2+1)}} = 0,$$

for $x \in [-2, -1]$. Observe that f is differentiable on $[-2, -1]$, and the derivative of f from the left on $x = -1$ is $\frac{3}{2}$. The function f is increasing and convex on $[-2, -1]$, and we may take $g(x) = \frac{1}{6}\left(x + 5 \arcsin \frac{x-1}{\sqrt{2(x^2+1)}}\right)$. An elementary calculation shows that $f(-2) < 0$ and $g(-2) \in [-2, -1]$; and therefore the hypotheses of Corollary 3.1 are verified. The procedure (3.1) leads to the following results:

n	x_n	$g(x_n)$	$f(x_n)$
0	-2.000000000000000	-1.37420481033188	$-7.853\,981\,633\,974\,48 \cdot 10^{-01}$
1	-1.406051288716128	-1.40401615840899	$-7.509\,542\,276\,017\,46 \cdot 10^{-01}$
2	-1.404223647476550	-1.40422359726392	$-2.442\,156\,368\,565\,04 \cdot 10^{-03}$
3	-1.404223602391970	-1.40422360239197	$-6.025\,515\,505\,460\,58 \cdot 10^{-08}$
4	-1.404223602391970	-1.40422360239197	$-3.718\,813\,451\,625\,28 \cdot 10^{-17}$

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