Asymptotic behaviour of intermediate points in certain mean-value theorems

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2000 Mathematics Subject Classification. 26A24, 41A60, 41A80.

Let $I$ be an open interval in $\mathbb{R}$, let $a \in I$, let $n \in \mathbb{N}$, and let $f : I \to \mathbb{R}$ be a function whose derivative $f^{(n)}$ exists on $I$. Then for any other point $x$ in $I$ one can expand $f(x)$ about the point $a$ up to $n$th power by the Lagrange–Taylor formula to obtain

$$f(x) = T_{n-1}(f; a)(x) + \frac{f^{(n)}(\xi)}{n!} (x - a)^n,$$

where $T_{n-1}(f; a)$ denotes the Taylor polynomial of degree $n - 1$ associated to $f$ at $a$. In (1) the intermediate point (or points) $\xi$ lies strictly between $a$ and $x$. In the special case when $n = 1$, formula (1) becomes the classical (Lagrange) mean value theorem

$$f(x) - f(a) = f'(\xi)(x - a).$$

In the last three decades there was some interest in the asymptotic behaviour of the intermediate point $\xi = \xi(x)$ in (1), (2) and other mean value theorems, when $x \to a$. Thus, A. G. Azpeitia [Amer. Math. Monthly 89 (1982), 311–312] proved that given $p \in \mathbb{N}$, the point $\xi$ in (1) satisfies

$$\xi = \xi(x) = a + \left(\frac{n + p}{n}\right)^{-1/p} (x - a) + o(|x - a|) \quad (x \to a).$$

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if \( f^{(n+p)} \) exists on the whole interval \( I \) and is continuous at \( a \) with \( f^{(n+j)}(a) = 0 \) (\( 1 \leq j < p \)) and \( f^{(n+p)}(a) \neq 0 \). This result was generalized by U. Abel [Amer. Math. Monthly 110 (2003), 627–633]. He derived for \( \xi \) a complete asymptotic expansion of the form

\[
\xi = \xi(x) = a + \sum_{k=1}^{\infty} \frac{c_k}{k!} (x-a)^k \quad (x \to a),
\]

provided that \( f \) possesses derivatives of sufficiently high order at \( a \).

A well-known generalization of (2) is the Cauchy mean value theorem: consider two functions \( f, g : I \to \mathbb{R} \) such that the derivatives \( f' \) and \( g' \) exist both on \( I \). If \( g' \) does not vanish in \( I \), then for every \( x \neq a \) in \( I \) one has

\[
g'(\xi)[f(x) - f(a)] = f'(\xi)[g(x) - g(a)],
\]

with intermediate point (or points) \( \xi \) strictly between \( a \) and \( x \). In a recent paper, D. I. Duca and O. Pop [Math. Inequal. Appl. 9 (2006), 375–389] proved that given \( p \in \mathbb{N} \), the point \( \xi \) in (3) satisfies

\[
\xi = \xi(x) = a + \frac{1}{\sqrt[p]{p+1}} (x-a) + o(|x-a|) \quad (x \to a)
\]

whenever the derivatives \( f^{(p+1)} \) and \( g^{(p+1)} \) exist on \( I \) and are both continuous at \( a \), \( f^{(j)}(a)g'(a) = f'(a)g^{(j)}(a) \) for all \( j \in \{2 \ldots p\} \), and \( f^{(p+1)}(a)g'(a) \neq f'(a)g^{(p+1)}(a) \).

In our talk we present similar asymptotic expansions for the intermediate points in other mean value theorems: the Cauchy–Taylor mean value theorem (which is a common generalization of (1), (2) and (3)), a generalization due to I. Pawlikowska of Flett’s mean value theorem, and a Cauchy version of Pawlikowska’s mean value theorem.