JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 47 (2018) no. 2, pp. 124–134
ictp.acad.ro/jnaat

A STANCU TYPE EXTENSION OF CHENEY AND SHARMA OPERATORS

TUĞBA BOSTANCI and GÜLEN BAŞCANBAZ-TUNCA

Abstract. In this paper, we introduce a Stancu type extension of the well known Cheney and Sharma operators. We consider a recurrence relation for the moments of the operators and give a local approximation result via suitable K-functional. Moreover, we show that each operator preserves the Lipschitz constant and order of a given Lipschitz continuous function.

MSC 2010. 41A36, 41A25.
Keywords. Stancu operator, Cheney-Sharma operator, Lipschitz continuous function, Abel-Jensen equalities.

1. INTRODUCTION

Let β be a nonnegative real number and consider the following Abel-Jensen formulas

$$(1)\qquad (u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} u\,(u+k\beta)^{k-1}\,[v+(m-k)\,\beta]^{m-k},$$

$$(2)\qquad (u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} (u+k\beta)^{k}\,v\,[v+(m-k)\,\beta]^{m-k-1},$$

$$(3)\qquad (u+v)\,(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u\,(u+k\beta)^{k-1}\,v\,[v+(m-k)\,\beta]^{m-k-1},$$

where $u,v\in\mathbb{R}$ and $m\geq 1$ (see, *e.g.*, [12]). In [5], Cheney and Sharma generalized the well-known Bernstein polynomials by taking $\beta\geq 0$, $u=x$ and $v=1-x$, $x\in[0,1]$, and $m=n\in\mathbb{N}$ in (1) and (3) as in the following forms:

$$(4)\qquad \mathbf{P}_n^{\beta}(f;x) := (1+n\beta)^{-n}\sum_{k=0}^{n}\binom{n}{k} x\,(x+k\beta)^{k-1}\,[1-x+(n-k)\,\beta]^{n-k}\, f\left(\tfrac{k}{n}\right)$$

and

$$(5)\qquad G_n^{\beta}(f;x) := \sum_{k=0}^{n} P_{n,k}^{\beta}(x)\, f\left(\tfrac{k}{n}\right),$$

*Department of Mathematics, University of Ankara, Turkey, 06100, e-mail: tbostanci@ankara.edu.tr, tunca@science.ankara.edu.tr.

where

$$P_{n,k}^{\beta}(x) := \frac{\binom{n}{k} x(x+k\beta)^{k-1}(1-x)[1-x+(n-k)\beta]^{n-k-1}}{(1+n\beta)^{n-1}} \tag{6}$$

for $f \in C[0,1]$, the space of real valued, continuous functions on $[0,1]$. Denoting $e_v(t) := t^v,\ t \in [0,1],\ v = 0,1,2,\cdots$, it is obvious that [5]

$$G_n^{\beta}(e_0;x) = 1. \tag{7}$$

Moreover, from [10], we have

$$G_n^{\beta}(e_1;x) = x. \tag{8}$$

Since $\beta \geq 0$, these operators are linear and positive and called as Bernstein type Cheney and Sharma operators. In [5], using the reduction formula

$$S(k,n,x,y) = xS(k-1,n,x,y) + n\beta S(k,n-1,x+\beta,y),$$

where S is given by

$$S(k,n,x,y) := \sum_{v=0}^{n} \binom{n}{v} (x+v\beta)^{v+k-1} (y+(n-v)\beta)^{n-v}, \tag{9}$$

the authors proved uniform convergence of each sequence of operators $\mathbf{P}_n^{\beta}(f)$ and $G_n^{\beta}(f)$ to f on $[0,1]$ by taking β as a sequence of nonnegative real numbers satisfying $\beta = o(\frac{1}{n}),\ n \to \infty$ in (4) and (5) (see [5, Lemma 1], also, [1, pp. 322-326]). Some works concerning these operators are [10], [6], [12], [11], [13], [14] and [2]. It is obvious that $\mathbf{P}_n^0 = G_n^0 = B_n$, where B_n is the n-th Bernstein operator.

In [8], Stancu constructed the following Bernstein type linear positive operators

$$L_{n,r}(f;x) := \sum_{k=0}^{n-r} P_{n-r,k}(x) \left[(1-x) f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right], \tag{10}$$

for $f \in C[0,1]$, where $P_{n,k}(x) := \binom{n}{k} x^k (1-x)^{n-k},\ n \in \mathbb{N},\ r$ is a non-negative integer parameter with $n > 2r$ (see, also, [9]).

In [15], Yang, Xiong and Cao extended the operators $L_{n,r}$ given by (10) to the multivariate setting on a simplex and called them as multivariate Stancu operators. In the work, using elementary method, the authors proved that the multivariate Stancu operators preserves Lipschitz property of the operand. In [4], Bustamante and Quesada gave an asymptotic property for Stancu operators $L_{n,r}$ related to Voronovskaja-type formula.

In the present paper, we consider Stancu operators $L_{n,r}$ in the basis of the Bernstein type Cheney and Sharma operators G_n^{β} given by (5). For this purpose, we consider

$$L_{n,r}^{\beta}(f;x) := \sum_{k=0}^{n-r} P_{n-r,k}^{\beta}(x) \left[(1-x) f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right], \tag{11}$$

for $f \in C[0,1]$ and r is a non-negative integer parameter with $n > 2r,\ n \in \mathbb{N}$, where $P^{\beta}_{n-r,k}$ is given by (6) with $n-r$ in places of n. We shall call these operators as Stancu type extension of Cheney and Sharma operators. For the calculation of moments, we use the same recurrence relationship which is obtained from another quantity that is slightly different from (9). Namely, the quantity (9) is closely related to (1), whereas the quantity that we shall use is related with (3). We study local approximation with the help of suitable $K-$functional, and show the preservation of Lipschitz' constant and order of a Lipschitz continuous function by $L^{\beta}_{n,r}$. To get approximation results, as in [5], we take β, as a sequence of positive real numbers such that $\beta = o(\frac{1}{n})$ $(n \to \infty)$. It is obvious that $L^{0}_{n,r}$ reduces to the Stancu operator $L_{n,r}$ given by (10) and $L^{\beta}_{n,0}$ reduces to the Cheney and Sharma operator (5).

2. AUXILIARY RESULTS

Using the similar technique of [5], we consider the following quantity $T(k,n,x,y)$ to get the subsequent recurrence relationship.

LEMMA 2.1. *Let $x, y \in [0,1],\ n \in \mathbb{N},\ k = 0,1,...,n,$ and*

$$T(k,n,x,y) := \sum_{v=0}^{n} \binom{n}{v} (x+v\beta)^{v+k-1} (y+(n-v)\beta)^{n-v-1}. \tag{12}$$

Then one has

$$T(k,n,x,y) = xT(k-1,n,x,y) + n\beta T(k,n-1,x+\beta,y). \tag{13}$$

Namely, T satisfies the same reduction formula that (9) holds.

Proof. Direct calculation gives the result. □

CONCLUSION 2.1. *For the quantity (12), one has the following results:*
(i) From (3), with $u = x,\ v = y$ and $m = n$, it readily follows that

$$xyT(0,n,x,y) = (x+y)(x+y+n\beta)^{n-1}.$$

(ii) From (2), with $u = x,\ v = y$ and $m = n$, it holds

$$yT(1,n,x,y) = (x+y+n\beta)^{n}.$$

(iii) Recursive application of the formula (13) gives that

$$T(2,n,x,y) = \sum_{v=0}^{n} \binom{n}{v} v!\,(x+v\beta)\,\beta^{v} T(1,n-v,x+v\beta,y).$$

Moreover, as in [5], using the fact

$$v! = \int_0^{\infty} e^{-s} s^{v} ds$$

and the binomial formula, one gets

(14)
$$yT(2,n,x,y)=\int_0^\infty e^{-s}\left[x(x+y+n\beta+s\beta)^n+ns\beta^2(x+y+n\beta+s\beta)^{n-1}\right]ds.$$

Below, making use of (12) and (13), we give $G_n^\beta(e_2;x)$.

LEMMA 2.2. *For every $x\in[0,1]$, $n\in\mathbb{N}$, one has*

$$G_n^\beta(e_2;x)=\tfrac{n-1}{n}\left\{x(x+2\beta)A_n+x(n-2)\beta^2B_n\right\}+\tfrac{x}{n},$$

where

$$A_n=\tfrac{1}{(1+n\beta)}\int_0^\infty e^{-s}\left(1+\tfrac{s\beta}{1+n\beta}\right)^{n-2}ds \tag{15}$$

and

$$B_n=\tfrac{1}{(1+n\beta)^2}\int_0^\infty se^{-s}\left(1+\tfrac{s\beta}{1+n\beta}\right)^{n-3}ds. \tag{16}$$

Proof. It is easy to see that

$$G_n^\beta(e_2;x)=\tfrac{n-1}{n}\tfrac{x(1-x)}{(1+n\beta)^{n-1}}T(2,n-2,x+2\beta,1-x)+\tfrac{x}{n}.$$

Using (14), $G_n^\beta(e_2;x)$ can be represented as

$$\begin{aligned}G_n^\beta(e_2;x)&=\\&=\tfrac{n-1}{n}\tfrac{x}{(1+n\beta)^{n-1}}\int_0^\infty e^{-s}\left[(x+2\beta)(1+n\beta+s\beta)^{n-2}+(n-2)s\beta^2(1+n\beta+s\beta)^{n-3}\right]ds+\tfrac{x}{n}\\&=\tfrac{n-1}{n}\left\{x(x+2\beta)A_n+x(n-2)\beta^2B_n\right\}+\tfrac{x}{n},\end{aligned}$$

where A_n and B_n are given by (15) and (16), respectively. □

Now, we need to evaluate the limits $\lim_{n\to\infty}A_n$ and $\lim_{n\to\infty}B_n$, when $\beta=o(\frac{1}{n})$.

LEMMA 2.3. *Let β be a sequence of positive real numbers such that $\beta=o(\frac{1}{n})$. Then we have*

$$\lim_{n\to\infty}A_n=1 \text{ and } \lim_{n\to\infty}B_n=1.$$

Proof. From (15), writing A_{n+2} and making change of variable $u=\frac{s\beta}{1+n\beta}$ we have

$$A_{n+2}=\tfrac{1}{\beta}\int_0^\infty e^{-\frac{(1+(n+2)\beta)}{\beta}u}(1+u)^n\,du.$$

As in [9], using the inequalities

$$e^{nu}\left(1-nu^2\right) \leqslant (1+u)^n \leqslant e^{nu} \tag{17}$$

it follows that

$$\tfrac{1}{1+2\beta} - \tfrac{2n\beta^2}{(1+2\beta)^3} \leqslant A_{n+2} \leqslant \tfrac{1}{1+2\beta}.$$

Similarly, from (16), for B_{n+3} we get

$$B_{n+3} = \tfrac{1}{\beta^2}\int_0^\infty e^{-\frac{(1+(n+3)\beta)}{\beta}u} u\,(1+u)^n\,du.$$

In view of (17), we obtain

$$\tfrac{1}{(1+3\beta)^2} - \tfrac{6n\beta^2}{(1+3\beta)^4} \leqslant B_{n+3} \leqslant \tfrac{1}{(1+3\beta)^2}.$$

Hence, using the fact $\beta = \beta_n > 0$ $(n \in \mathbb{N})$ satisfying $\lim_{n\to\infty} n\beta_n = 0$, we conclude that $\lim_{n\to\infty} A_n = 1$ and $\lim_{n\to\infty} B_n = 1$. □

3. APPROXIMATION PROPERTIES OF THE STANCU TYPE EXTENSION OF CHENEY AND SHARMA OPERATORS

In this section, we study some approximation properties of the Stancu type extension of Cheney and Sharma operators $L^{\beta}_{n,r}$ given by (11). The moments of the operators can be expressed in terms of the moments of the Cheney and Sharma operators G^{β}_{n}. Namely, we have

Lemma 3.1. *For every* $x \in [0,1]$, $n > 2r$, $r \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, *we have*

$$\begin{aligned} L^{\beta}_{n,r}(e_0;x) &= 1,\\ L^{\beta}_{n,r}(e_1;x) &= x,\\ L^{\beta}_{n,r}(e_2;x) &= \tfrac{(n-r)(n-r-1)}{n^2}\left\{x(x+2\beta)A_{n-r} + x(n-r-2)\beta^2 B_{n-r}\right\}\\ &\quad + \tfrac{x}{n^2}\left\{(n-r)(1+2xr)+r^2\right\}, \end{aligned}$$

where A_{n-r} *and* B_{n-r} *are given by* (15) *and* (16), *respectively.*

Proof. Taking (5), (7), (8), (11) and Lemma 2.2 into consideration, we get

$$\begin{aligned} L^{\beta}_{n,r}(e_0;x) &= G^{\beta}_{n-r}(e_0;x) = 1,\\ L^{\beta}_{n,r}(e_1;x) &= \tfrac{n-r}{n}G^{\beta}_{n-r}(e_1;x) + \tfrac{rx}{n}G^{\beta}_{n-r}(e_0;x) = \left(1-\tfrac{r}{n}\right)x + \tfrac{r}{n}x = x,\\ L^{\beta}_{n,r}(e_2;x) &= \left(\tfrac{n-r}{n}\right)^2 G^{\beta}_{n-r}(e_2;x) + 2xr\tfrac{n-r}{n^2}G^{\beta}_{n-r}(e_1;x) + \tfrac{xr^2}{n^2}G^{\beta}_{n-r}(e_0;x)\\ &= \tfrac{(n-r)(n-r-1)}{n^2}\left\{x(x+2\beta)A_{n-r} + x(n-r-2)\beta^2 B_{n-r}\right\}\\ &\quad + \tfrac{x}{n^2}\left\{(n-r)(1+2xr)+r^2\right\}, \end{aligned}$$

where A_{n-r} and B_{n-r} are given by (15) and (16). □

Thus, from Lemma 3.1 one easily obtains the following result.

Corollary 3.1. *For every* $x \in [0,1]$, $n > 2r$, $r \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, *we have*

$$\begin{aligned} L_{n,r}^{\beta}\left((e_1 - x); x\right) &= 0, \\ L_{n,r}^{\beta}\left((e_1 - x)^2; x\right) &= \left(\tfrac{n-r}{n}\right)^2 \tfrac{n-r-1}{n-r} \left\{x(x+2\beta)A_{n-r} + x(n-r-2)\beta^2 B_{n-r}\right\} \\ &\quad + \tfrac{2x^2 r(n-r) + xr^2}{n^2} - x^2 \\ &=: \delta_n(x). \end{aligned}$$

Let us denote the uniform norm on $C[0,1]$ by $\|.\|$. It is not difficult to show that the operators $L_{n,r}^{\beta}$ are bounded from $C[0,1]$ onto itself:

Lemma 3.2. *For every* $f \in C[0,1]$ *we have* $\|L_{n,r}^{\beta}(f)\| \leq \|f\|$.

Proof.

$$\begin{aligned} \left|L_{n,r}^{\beta}(f;x)\right| &= \left|\sum_{k=0}^{n-r} P_{n-r,k}^{\beta}(x)\left[(1-x)f\left(\tfrac{k}{n}\right) + xf\left(\tfrac{k+r}{n}\right)\right]\right| \\ &\leq \sum_{k=0}^{n-r} P_{n-r,k}^{\beta}(x)\left|(1-x)f\left(\tfrac{k}{n}\right) + xf\left(\tfrac{k+r}{n}\right)\right| \\ &\leq \sum_{k=0}^{n-r} P_{n-r,k}^{\beta}(x)\left\{(1-x)\left|f\left(\tfrac{k}{n}\right)\right| + x\left|f\left(\tfrac{k+r}{n}\right)\right|\right\} \\ &\leq \|f\| \sum_{k=0}^{n-r} P_{n-r,k}^{\beta}(x)\{1-x+x\} \\ &= \|f\|. \end{aligned}$$

□

Recall that the Peetre K-functional is defined as

$$K_2(f,\delta) = \inf\left\{\|f-g\| + \delta\|g''\| : g \in W^2\right\},$$

where $\delta > 0$, $W^2 := \{g \in C[0,1] : g', g'' \in C[0,1]\}$. From p.177, Theorem 2.4 of [7], there is a positive constant $C > 0$ such that

$$K_2(f,\delta) \leq C\omega_2\left(f, \sqrt{\delta}\right), \tag{18}$$

where

$$\omega_2(f,\delta) = \sup_{0 \leq h \leq \delta} \sup_{x,\, x+2h \in [0,1]} |f(x+2h) - 2f(x+h) + f(x)|$$

is the second order modulus of smoothness of $f \in C[0,1]$.

Theorem 3.1. *Let* $f \in C[0,1]$, $x \in [0,1]$, $n > 2r$, $r \in \mathbb{N} \cup \{0\}$, $n \in \mathbb{N}$, *and* β *be a sequence of positive real numbers such that* $\beta = o\left(\frac{1}{n}\right)$. *Then*

$$\left|L_{n,r}^{\beta}(f;x) - f(x)\right| \leq C\omega_2\left(f, \sqrt{\delta_{n,r}(x)}\right),$$

where $\delta_{n,r}(x)$ *is the same as in Corollary* 3.1 *and* C *is a positive constant.*

Proof. For any function $g \in W^2$ and $x, t \in [0,1]$, the Taylor formula gives that

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-y) g''(y)\, dy.$$

Applying $L_{n,r}^{\beta}$ on both sides of the above formula, linearity and Lemma 3.1 give that

$$\begin{aligned}\left|L_{n,r}^{\beta}(g;x) - g(x)\right| &\leq \|g''\| L_{n,r}^{\beta}\left(\int_x^t |t-y|\, dy; x\right) \leq \|g''\| L_{n,r}^{\beta}\left((t-x)^2; x\right) \\ &= \|g''\| \delta_{n,r}(x).\end{aligned}$$

Therefore,

$$\begin{aligned}\left|L_{n,r}^{\beta}(f;x) - f(x)\right| &\leq \left|L_{n,r}^{\beta}(f-g;x) - (f-g)(x)\right| + \left|L_{n,r}^{\beta}(g;x) - g(x)\right| \\ &\leq 2\|f-g\| + \|g''\| \delta_{n,r}(x).\end{aligned}$$

Passing to the infimum over all $g \in W^2$ and taking (18) into consideration, we obtain

$$\left|L_{n,r}^{\beta}\ (f;x) - f(x)\right| \leq 2K_2(f, \delta_{n,r}(x)) \leq C\omega_2\left(f, \sqrt{\delta_{n,r}(x)}\right),$$

which completes the proof. □

Next result provides the property of the preservation of Lipschitz' constant and order of a Lipschitz continuous function by each $L_{n,r}^{\beta}$. The same result for the Bernstein polynomials was proved by Brown, Elliott and Paget [3], also, for the Cheney and Sharma operators G_n^{β} was obtained in [2] and for the multivariate Stancu operators was proved in [15].

Recall that the class $\mathrm{Lip}_M(\alpha, [0,1])$ and the convexity of $f \in C[0,1]$ are defined, respectively, as

$$\begin{aligned}&\mathrm{Lip}_M(\alpha, [0,1]) := \\ &:= \{f \in C[0,1] : |f(x) - f(y)| \leq M|x-y|^{\alpha}\ \forall x, y \in [0,1],\ 0 < \alpha \leq 1\}\end{aligned}$$

and

$$f \text{ is convex} \Leftrightarrow\ f\left(\sum_{k=1}^{n} \alpha_k x_k\right) \leq \sum_{k=1}^{n} \alpha_k f(x_k), \quad \forall x_1, x_2, \dots, x_n \in [0,1],$$

where $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$ satisfying $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$.

THEOREM 3.2. *Let* $f \in \mathrm{Lip}_M(\alpha, [0,1])$. *Then* $L_{n,r}^{\beta}(f) \in \mathrm{Lip}_M(\alpha, [0,1])$ *for all* $n \in \mathbb{N}$.

Proof. Assume that $x, y \in [0,1]$ satisfy $y \geq x$. Following similar steps used in [2], from (11) and (3) $L_{n,r}^{\beta}(f;y)$ can be written as

$$
\begin{aligned}
L_{n,r}^{\beta}\left(f;y\right) = & \tfrac{1}{\left(1+\left(n-r\right)\beta\right)^{n-r-1}}\times \\
& \times\sum_{j=0}^{n-r}\sum_{k=0}^{j}\tbinom{n-r}{j}\tbinom{j}{k}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+\left(j-k\right)\beta\right]^{j-k-1} \\
& \times\left(1-y\right)\left[1-y+\left(n-r-j\right)\beta\right]^{n-r-j-1} \\
& \times\left\{\left(1-y\right)f\left(\tfrac{j}{n}\right)+yf\left(\tfrac{j+r}{n}\right)\right\}.
\end{aligned}
$$

Changing the order of the summations and letting $j-k=l$ in the result, we obtain

(19)

$$
\begin{aligned}
L_{n,r}^{\beta}\left(f;y\right) = & \tfrac{1}{\left(1+\left(n-r\right)\beta\right)^{n-r-1}}\times \\
& \times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{\left(n-r\right)!}{\left(n-r-k-l\right)!k!l!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1} \\
& \times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1} \\
& \times\left\{\left(1-y\right)f\left(\tfrac{k+l}{n}\right)+yf\left(\tfrac{k+l+r}{n}\right)\right\}.
\end{aligned}
$$

Now, consider the case $L_{n,r}^{\beta}\left(f;x\right)$. Taking $u=y-x$, $v=1-y$ and $m=n-r-k$ in (3), we can write $L_{n,r}^{\beta}\left(f;x\right)$ as

(20)

$$
\begin{aligned}
L_{n,r}^{\beta}\left(f;x\right) = & \tfrac{1}{\left(1+\left(n-r\right)\beta\right)^{n-r-1}}\times \\
& \times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{\left(n-r\right)!}{k!l!\left(n-r-k-l\right)!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1} \\
& \times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1} \\
& \times\left\{\left(1-x\right)f\left(\tfrac{k}{n}\right)+xf\left(\tfrac{k+r}{n}\right)\right\}.
\end{aligned}
$$

Thus, subtracting (20) from (19) we get

$$
\begin{aligned}
L_{n,r}^{\beta}\left(f;y\right)-L_{n,r}^{\beta}\left(f;x\right) = & \tfrac{1}{\left(1+\left(n-r\right)\beta\right)^{n-r-1}}\times \\
& \times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{\left(n-r\right)!}{k!l!\left(n-r-k-l\right)!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1} \\
& \times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1} \\
& \times\left\{\left(1-y\right)f\left(\tfrac{k+l}{n}\right)+yf\left(\tfrac{k+l+r}{n}\right)-\left(1-x\right)f\left(\tfrac{k}{n}\right)-xf\left(\tfrac{k+r}{n}\right)\right\}.
\end{aligned}
$$

Adding and dropping the terms $yf(\frac{k}{n})$ and $xf(\frac{k+l+r}{n})$, the above formula takes to the following form:

$$\begin{aligned}
&L_{n,r}^{\beta}\left(f;y\right)-L_{n,r}^{\beta}\left(f;x\right)=\\
&=\tfrac{1}{(1+(n-r)\beta)^{n-r-1}}\times\\
&\quad\times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{(n-r)!}{k!l!(n-r-k-l)!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1}\\
&\quad\times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1}\\
&\quad\times\left\{\left(1-y\right)\left[f\left(\tfrac{k+l}{n}\right)-f\left(\tfrac{k}{n}\right)\right]+x\left[f\left(\tfrac{k+l+r}{n}\right)-f\left(\tfrac{k+r}{n}\right)\right]\right.\\
&\quad\left.+\left(y-x\right)\left[f\left(\tfrac{k+l+r}{n}\right)-f\left(\tfrac{k}{n}\right)\right]\right\}.
\end{aligned}$$

Therefore, taking the absolute values of both sides of the last formula, and using the hypothesis that $f\in\mathrm{Lip}_M\left(\alpha,[0,1]\right)$ in the result, we get

$$\begin{aligned}
&\left|L_{n,r}^{\beta}\left(f;y\right)-L_{n,r}^{\beta}\left(f;x\right)\right|\leq\\
&\leq\tfrac{1}{(1+(n-r)\beta)^{n-r-1}}\times\\
&\quad\times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{(n-r)!}{k!l!(n-r-k-l)!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1}\\
&\quad\times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1}\\
&\quad\times\left\{\left(1-y\right)\left|f\left(\tfrac{k+l}{n}\right)-f\left(\tfrac{k}{n}\right)\right|+x\left|f\left(\tfrac{k+l+r}{n}\right)-f\left(\tfrac{k+r}{n}\right)\right|\right.\\
&\quad\left.+\left(y-x\right)\left|f\left(\tfrac{k+l+r}{n}\right)-f\left(\tfrac{k}{n}\right)\right|\right\}\\
&\leq\tfrac{M}{(1+(n-r)\beta)^{n-r-1}}\times\\
&\quad\times\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tfrac{(n-r)!}{k!l!(n-r-k-l)!}x\left(x+k\beta\right)^{k-1}\left(y-x\right)\left[y-x+l\beta\right]^{l-1}\\
&\quad\times\left(1-y\right)\left[1-y+\left(n-r-k-l\right)\beta\right]^{n-r-k-l-1}\\
&\quad\times\left\{\left(1-\left(y-x\right)\right)\left(\tfrac{l}{n}\right)^{\alpha}+\left(y-x\right)\left(\tfrac{l+r}{n}\right)^{\alpha}\right\}
\end{aligned}$$

by the assumption that $x\leq y$. Changing the order of the summations, the above formula gives that

$$\begin{aligned}
&\left|L_{n,r}^{\beta}\left(f;y\right)-L_{n,r}^{\beta}\left(f;x\right)\right|\leq\\
&\leq\tfrac{M}{(1+(n-r)\beta)^{n-r-1}}\sum_{l=0}^{n-r}\tbinom{n-r}{l}\left(y-x\right)\left[y-x+l\beta\right]^{l-1}\times
\end{aligned}$$

$$\times \sum_{k=0}^{n-r-l} \tbinom{n-r-l}{k} x\,(x+k\beta)^{k-1}\,(1-y)\,[1-y+(n-r-l-k)\,\beta]^{n-r-l-k-1}$$
$$\times \left\{(1-(y-x))\left(\tfrac{l}{n}\right)^{\alpha}+(y-x)\left(\tfrac{l+r}{n}\right)^{\alpha}\right\}.$$

By taking $u=x$, $v=1-y$ and $m=n-r-l$ in (3), this formula can be simplified as

$$\begin{aligned}
&\left|L_{n,r}^{\beta}(f;y)-L_{n,r}^{\beta}(f;x)\right| \le \\
&\le \tfrac{M}{(1+(n-r)\beta)^{n-r-1}} \sum_{l=0}^{n-r} \tbinom{n-r}{l}\,(y-x)\,[y-x+l\beta]^{l-1}\times \\
&\quad\times (1-(y-x))\,[1-(y-x)+(n-r-l)\,\beta]^{n-r-l-1} \\
&\quad\times \left\{(1-(y-x))\left(\tfrac{l}{n}\right)^{\alpha}+(y-x)\left(\tfrac{l+r}{n}\right)^{\alpha}\right\}.
\end{aligned}$$

Here, taking $x_1=\frac{l}{n}$, $x_2=\frac{l+r}{n}$, and regarding the nonnegative constants α_1, α_2 as $\alpha_1=y-x$ and $\alpha_2=1-(y-x)$ that satisfy $\alpha_1+\alpha_2=1$, and using the fact that $g(t)=t^{\alpha}$, $0<\alpha\le 1$, is concave, then the last formula reduces to

$$\begin{aligned}
&\left|L_{n,r}^{\beta}(f;y)-L_{n,r}^{\beta}(f;x)\right| \le \\
&\le \tfrac{M}{(1+(n-r)\beta)^{n-r-1}} \sum_{l=0}^{n-r} \tbinom{n-r}{l}\,(y-x)\,[y-x+l\beta]^{l-1}\times \\
&\quad\times (1-(y-x))\,[1-(y-x)+(n-r-l)\,\beta]^{n-r-l-1} \\
&\quad\times \left\{(1-(y-x))\,\tfrac{l}{n}+(y-x)\left(\tfrac{l+r}{n}\right)\right\}^{\alpha} \\
&= M\sum_{l=0}^{n-r} P_{n-r}^{\beta}(y-x)\left\{(1-(y-x))\,\tfrac{l}{n}+(y-x)\left(\tfrac{l+r}{n}\right)\right\}^{\alpha},
\end{aligned}$$

where P_{n-r}^{β} is given by (6). Here, the case $\alpha=1$ is obvious. For the case $0<\alpha<1$; application of Hölder's inequality, with conjugate pairs $p=\frac{1}{\alpha}$ and $q=\frac{1}{1-\alpha}$, leads to

$$\begin{aligned}
&\left|L_{n,r}^{\beta}(f;y)-L_{n,r}^{\beta}(f;x)\right| \le \\
&\le M\left\{\sum_{l=0}^{n-r} P_{n-r}^{\beta}(y-x)\left[(1-(y-x))\,\tfrac{l}{n}+(y-x)\left(\tfrac{l+r}{n}\right)\right]\right\}^{\alpha}\left\{\sum_{l=0}^{n-r} P_{n-r}^{\beta}(y-x)\right\}^{1-\alpha} \\
&= M\,\left\{L_{n,r}^{\beta}(e_1;y-x)\right\}^{\alpha}\left\{L_{n,r}^{\beta}(e_0;y-x)\right\}^{1-\alpha} \\
&= M(y-x)^{\alpha},
\end{aligned}$$

by Lemma 3.1, which completes this proof. $\square$

REFERENCES

[1] F. Altomare, M. Campiti, *Korovkin-Type Approximaton Theory and Its Applications*, Walter de Gruyter, Berlin-New York, 1994.

[2] G. Başcanbaz-Tunca, A. Erençin, F. Taşdelen, *Some properties of Bernstein type Cheney and Sharma Operators,* General Mathematics, **24** (2016) nos. 1–2, pp. 17–25.

[3] B.M. Brown, D. Elliott, D.F. Paget, *Lipschitz constants for the Bernstein polynomials of a Lipschitz continuous function*, J. Approx. Theory, **49** (1987) no. 2, pp. 196–199.

[4] J. Bustamante, J.M. Quesada, *A property of Ditzian-Totik second order moduli*, Appl. Math. Lett. **23** (2010), no. 5, pp. 576–580.

[5] E.W. Cheney, A. Sharma, *On a generalization of Bernstein polynomials*, Riv. Mat. Univ. Parma, **2**(5) (1964), pp. 77–84.

[6] M. Crăciun, *Approximation operators constructed by means of Sheffer sequences*, Rev. Anal. Numér. Théor. Approx., **30** (2001) no. 2, pp. 135–150.

[7] R. DeVore, G.G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.

[8] D.D. Stancu, *Quadrature formulas constructed by using certain linear positive operators,* Numerical Integration (Proc. Conf., Oberwolfach, 1981), ISNM 57 (1982), pp. 241–251, Birkhäuser Verlag, Basel.

[9] D.D. Stancu, *Approximation of functions by means of a new generalized Bernstein operator,* Calcolo, **20** (1983) no. 2, pp. 211–229.

[10] D.D. Stancu, C. Cismaşiu,*On an approximating linear positive operator of Cheney-Sharma*, Rev. Anal. Numér. Théor. Approx., **26** (1997), nos. 1–2, pp. 221–227.

[11] D.D. Stancu, *Use of an identity of A. Hurwitz for construction of a linear positive operator of approximation*, Rev. Anal. Numér. Théor. Approx., **31** (2002) no. 1, pp. 115–118.

[12] D.D. Stancu, E.I. Stoica, *On the use of Abel-Jensen type combinatorial formulas for construction and investigation of some algebraic polynomial operators of approximation*, Stud. Univ. Babeş-Bolyai Math., **54** (2009), no. 4, pp. 167–182.

[13] D.D. Stancu, M.R. Occorsio, *On Approximation by binomial operators of Tiberiu Popoviciu type*, Rev. Anal. Numér. Théor. Approx., 27 (1998), pp. 167–181.

[14] T. Cătinaş, D. Otrocol, *Iterates of multivariate Cheney-Sharma operators.* J. Comput. Anal. Appl., **15** (2013), no. 7, pp. 1240–1246.

[15] R. Yang, J. Xiong, F. Cao, *Multivariate Stancu operators defined on a simplex,* Appl. Math. Comput., **138** (2003), pp. 189–198.

Received by the editors: September 22, 2017. Accepted: July 3, 2018. Published online: February 17, 2019.