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ON BERMAN'S PHENOMENON FOR (0,1,2) HERMITE–FEJÉR INTERPOLATION

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Abstract. Given $f \in C[-1,1]$ and n points (nodes) in $[-1,1]$, the *Hermite–Fejér interpolation* (HFI) polynomial is the polynomial of degree at most $2n-1$ which agrees with f and has zero derivative at each of the nodes. In 1916, L. Fejér showed that if the nodes are chosen to be the zeros of $T_n(x)$, the nth Chebyshev polynomial of the first kind, then the HFI polynomials converge uniformly to f as $n \to \infty$. Later, D. L. Berman established the rather surprising result that this convergence property is no longer true for all f if the Chebyshev nodes are augmented by including the endpoints -1 and 1 as additional nodes. This behaviour has become known as *Berman's phenomenon.* The aim of this paper is to investigate Berman's phenomenon in the setting of $(0,1,2)$ HFI, where the interpolation polynomial agrees with f and has vanishing first and second derivatives at each node. The principal result provides simple necessary and sufficient conditions, in terms of the (one-sided) derivatives of f at ± 1, for pointwise and uniform convergence of $(0,1,2)$ HFI on the augmented Chebyshev nodes if $f \in C^4[-1,1]$, and confirms that Berman's phenomenon occurs for $(0,1,2)$ HFI.

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1. INTRODUCTION

Suppose $f \in C[-1,1]$ and let

$$X = \{x_{k,n} : k = 0,1,2,\ldots,n-1;\ n = 1,2,3,\ldots\} \tag{1}$$

be an infinite triangular matrix of nodes such that, for all n,

$$1 \geq x_{0,n} > x_{1,n} > \ldots > x_{n-1,n} \geq -1. \tag{2}$$

The well-known Lagrange interpolation polynomial of f is the polynomial $L_n(X,f)(x) = L_n(X,f,x)$ of degree at most $n-1$ which satisfies

$$L_n(X,f,x_{k,n}) = f(x_{k,n}), \qquad 0 \leq k \leq n-1.$$

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A classic result due to Faber [6] states that for any X there exists $f \in C[-1,1]$ so that $L_n(X,f)$ does not converge uniformly to f on $[-1,1]$ as $n \to \infty$. On the other hand, a more positive result occurs for the matrix of Chebyshev nodes

$$T = \Big\{x_{k,n} = \cos\big(\tfrac{2k+1}{2n}\pi\big) : k = 0,1,2,\dots,n-1;\ n = 1,2,3,\dots\Big\}$$

where, for each n, the $x_{k,n}$ are the zeros of the nth Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$, $-1 \le x \le 1$. This result states that if the modulus of continuity $\omega(\delta; f)$ of f is defined by

$$\omega(\delta; f) = \max\{|f(s) - f(t)| : -1 \le s,t \le 1, |s-t| \le \delta\},$$

then $L_n(T,f)$ converges uniformly to f under the quite mild restriction

$$\omega(\tfrac{1}{n}; f)\log n \to 0, \text{ as } n \to \infty$$

(see Rivlin [11, Chapter 4] for details and references).

A generalization of Lagrange interpolation is provided by Hermite–Fejér interpolation (HFI). Given a non-negative integer m and nodes X defined by (1) and (2), the $(0,1,\dots,m)$ HFI polynomial $H_{m,n}(X,f)(x) = H_{m,n}(X,f,x)$ of f is the unique polynomial of degree at most $(m+1)n-1$ which satisfies the $(m+1)n$ conditions

$$\begin{cases} H_{m,n}(X,f,x_{k,n}) = f(x_{k,n}), & 0 \le k \le n-1, \\ H_{m,n}^{(r)}(X,f,x_{k,n}) = 0, & 1 \le r \le m,\ 0 \le k \le n-1. \end{cases}$$

Note that $H_{0,n}(X,f) = L_n(X,f)$. The original motivation for studying HFI was provided by Fejér [7], who in 1916 showed that if $f \in C[-1,1]$, then $\|H_{1,n}(T,f) - f\| \to 0$ as $n \to \infty$ (here and subsequently, $\|\cdot\|$ denotes the uniform norm on $[-1,1]$). Thus on the Chebyshev nodes, $(0,1)$ HFI succeeds where Lagrange interpolation may fail.

In the years since Fejér's work, $(0,1,\dots,m)$ HFI has been much studied by many authors. In this paper our focus is on an aspect of HFI that has become known as *Berman's phenomenon*, which occurs if the Chebyshev nodes are augmented by the end points of the interval $[-1,1]$. In other words, we will be studying $(0,1,\dots,m)$ HFI on the nodes $T_a = \{x_{k,n+2} : 0 \le k \le n+1, n = 1,2,3,\dots\}$, where

$$\begin{cases} x_{0,n+2} = 1, \quad x_{n+1,n+2} = -1, \\ x_{k,n+2} = \cos\big(\frac{(2k-1)\pi}{2n}\big), \quad 1 \le k \le n. \end{cases} \tag{3}$$

Thus, for each n, the $x_{k,n+2}$ are the zeros of $(1-x^2)T_n(x)$.

Initially it might be thought that augmenting the Chebyshev nodes with ± 1 will have little effect on the convergence behaviour of interpolation polynomials when compared with interpolation on the Chebyshev nodes alone. However, D. L. Berman [1] was able to show that if $f(x) = |x|$, then $H_{1,n}(T_a, f, 0)$ diverges, while later [2] he showed that for $g(x) = x^2$, $H_{1,n}(T_a, g, x)$ does not converge to x^2 at any point of $(-1,1)$. (This result for x^2 does not extend to $[-1,1]$ because ± 1 are interpolation nodes for all n.) An explanation for Berman's phenomenon was provided by Bojanic, as follows.

THEOREM 1 (Bojanic). [3]. *If $f \in C[-1,1]$ has left and right derivatives $f'_L(1)$ and $f'_R(-1)$ at 1 and -1, respectively, then $H_{1,n}(T_a, f)$ converges uniformly to f on $[-1,1]$ if and only if $f'_L(1) = f'_R(-1) = 0$.*

At this point it is natural to ask whether Berman's phenomenon occurs for $(0,1,\ldots,m)$ HFI if $m \neq 1$. When $m = 0$ (Lagrange interpolation) it is straightforward to show that it does *not* occur. This follows from the representation

$$\begin{aligned} L_{n+2}(T_a, f, x) =& L_n(T, f, x) + \tfrac{1}{2} T_n(x) \{ (1+x)[f(1) - L_{n-1}(T, f, 1)] \\ &+ (-1)^n (1-x)[f(-1) - L_{n-1}(T, f, -1)] \}, \end{aligned}$$

which can be verified by observing that both sides of the equation are polynomials of degree at most $n+1$ which agree at the $n+2$ nodes $x_{k,n+2}$ given by (3). Thus $L_n(T_a, f) \to f$ uniformly on $[-1,1]$ whenever $L_n(T, f) \to f$ uniformly on $[-1,1]$.

The question of whether Berman's phenomenon occurs for $(0,1,\ldots,m)$ HFI for any $m > 1$ was answered in the affirmative by Cook and Mills [5] in 1975, who showed that if $h(x) = (1-x^2)^3$, then $H_{3,n}(T_a, h, 0)$ diverges. (Incidentally, it was in [5] that the term *Berman's phenomenon* was first used.) The result of Cook and Mills was later extended by Maky [9] who showed that $H_{3,n}(T_a, h, x)$ diverges at each point in $(-1,1)$. These findings for $(0,1,2,3)$ HFI on T_a contrast with the earlier result of Krylov and Steuermann [8] that $H_{3,n}(T, f)$ converges uniformly to f on $[-1,1]$ for *any* $f \in C[-1,1]$.

In this paper our focus will be on $(0,1,2)$ HFI. Here it was shown by Szabados and Varma [13] that, as with Lagrange interpolation, for any matrix of nodes X there exists $f \in C[-1,1]$ so that $H_{2,n}(X, f)$ does not converge uniformly to f on $[-1,1]$. On the other hand, and again like Lagrange interpolation, it follows from Byrne et al. [4, Theorem 1] that if $\omega(1/n; f) \log n \to 0$ as $n \to \infty$, then $H_{2,n}(T, f)$ converges uniformly to f.

To investigate $(0,1,2)$ HFI on T_a it proves helpful to follow the approach of Bojanic [3], and introduce incremental modifications to the $(0,1,2)$ HFI process on T. To this end, with nodes $x_k = x_{k,n+2}$ defined by (3) and $f \in C[-1,1]$, define the polynomial $Q_{2,n+2}(T_a, f)$ of degree at most $3n+1$ by the $3n+2$ conditions

$$(4) \qquad \begin{cases} Q_{2,n+2}(T_a, f, x_k) = f(x_k), & 0 \leq k \leq n+1, \\ Q'_{2,n+2}(T_a, f, x_k) = Q''_{2,n+2}(T_a, f, x_k) = 0, & 1 \leq k \leq n, \end{cases}$$

and define the polynomial $R_{2,n+2}(T_a, f)$ of degree at most $3n+3$ by the $3n+4$ conditions

$$\left\{\begin{array}{ll} R_{2,n+2}(T_a,f,x_k) = f(x_k), R'_{2,n+2}(T_a,f,x_k) = 0, & 0 \le k \le n+1, \\ R''_{2,n+2}(T_a,f,x_k) = 0, & 1 \le k \le n. \end{array}\right. \tag{5}$$

Also, for future reference, recall that $H_{2,n+2}(T_a, f)$ is defined by the $3n+6$ conditions

$$\left\{\begin{array}{l} H_{2,n+2}(T_a,f,x_k) = f(x_k), \\ H'_{2,n+2}(T_a,f,x_k) = H''_{2,n+2}(T_a,f,x_k) = 0, \qquad 0 \le k \le n+1. \end{array}\right. \tag{6}$$

Our principal results are presented in the following theorem. Note that throughout this paper we are concerned with functions defined on $[-1,1]$, and so any derivative evaluated at 1 or -1 is assumed to be the appropriate one-sided derivative.

Theorem 2. *Suppose that the polynomials* $Q_{2,n+2}(T_a, f)$, $R_{2,n+2}(T_a, f)$ *and* $H_{2,n+2}(T_a, f)$ *are defined by* (4)–(6), *where the interpolation nodes* $x_k = x_{k,n+2}$ *are defined by* (3).

a) *If* $f \in C[-1,1]$ *and* $\lim_{n\to\infty} \|H_{2,n}(T,f) - f\| = 0$, *then*

$$\lim_{n\to\infty} \|Q_{2,n+2}(T_a,f) - f\| = 0.$$

b) *If* $f \in C^2[-1,1]$, *then* $\lim_{n\to\infty} \|R_{2,n+2}(T_a,f) - f\| = 0$ *if and only if* $f'(1) = f'(-1) = 0$. *Furthermore, if* $f'(1)$ *and* $f'(-1)$ *are not both* 0, *then* $R_{2,n+2}(T_a,f,x)$ *is divergent if* $0 < |x| < 1$, *and* $R_{2,n+2}(T_a,f,0)$ *converges to* $f(0)$ *if and only if* $f'(1) = f'(-1)$.

c) *If* $f \in C^4[-1,1]$, *then* $\lim_{n\to\infty} \|H_{2,n+2}(T_a,f) - f\| = 0$ *if and only if*

$$f'(1) = f'(-1) = f''(1) = f''(-1) = 0. \tag{7}$$

If $0 < |x| < 1$ *and* (7) *does not hold, then* $H_{2,n+2}(T_a,f,x)$ *is divergent; in particular, if* $f'(1)$ *and* $f'(-1)$ *are not both 0, then*

$$\limsup_{n\to\infty} \tfrac{1}{n^2}|H_{2,n+2}(T_a,f,x)| > 0. \tag{8}$$

Furthermore, $H_{2,n+2}(T_a,f,0)$ *converges to* $f(0)$ *if and only if* $f'(1) = f'(-1)$ *and* $f''(1) = -f''(-1)$ *(which occurs, for example, if* f *is odd). If* $f'(1) \ne f'(-1)$, *then*

$$\limsup_{n\to\infty} \tfrac{1}{n^2}|H_{2,n+2}(T_a,f,0)| > 0. \tag{9}$$

Theorem 2 confirms that Berman's phenomenon occurs for (0,1,2) HFI. In particular, the following results hold.

Corollary 3. $(0,1,2)$ *HFI provides the following illustrations of Berman's phenomenon.*

a) *If* $f(x) = x$, *then* $\limsup_{n\to\infty} \frac{1}{n^2}|H_{2,n+2}(T_a, f, x)| > 0$ *for* $0 < |x| < 1$ *and* $\lim_{n\to\infty} H_{2,n+2}(T_a, f, 0) = 0$.

b) *If* $g(x) = x^2$, *then* $\limsup_{n\to\infty} \frac{1}{n^2}|H_{2,n+2}(T_a, g, x)| > 0$ *for all* x *in* $(-1, 1)$.

The theorem will be proved *via* a sequence of lemmas in the next section.

2. PROOF OF THE MAIN RESULT (THEOREM 2)

We begin by noting that the polynomials $Q_{2,n+2}(T_a, f)$, $R_{2,n+2}(T_a, f)$ and $H_{2,n+2}(T_a, f)$, defined by (4)–(6), are related to each other and to $H_{2,n}(T, f)$ according to the formulas

$$\begin{aligned} &Q_{2,n+2}(T_a, f, x) = \\ &= H_{2,n}(T, f, x) + \tfrac{1}{2}\,(T_n(x))^3 \\ &\quad \times (1 + x)\,(f(1) - H_{2,n}(T, f, 1)) + (-1)^n(1 - x)\,(f(-1) - H_{2,n}(T, f, -1))]\,, \end{aligned} \tag{10}$$

$$\begin{aligned} &R_{2,n+2}(T_a, f, x) = \\ &= Q_{2,n+2}(T_a, f, x) + \tfrac{1}{4}\,(T_n(x))^3\,(1 - x^2) \\ &\quad \times \left[(1 + x)Q'_{2,n+2}(T_a, f, 1) - (-1)^n(1 - x)Q'_{2,n+2}(T_a, f, -1)\right] \end{aligned} \tag{11}$$

and

$$\begin{aligned} &H_{2,n+2}(T_a, f, x) = \\ &= R_{2,n+2}(T_a, f, x) - \tfrac{1}{16}\,(T_n(x))^3\,(1 - x^2)^2 \\ &\quad \times \left[(1 + x)R''_{2,n+2}(T_a, f, 1) + (-1)^n(1 - x)R''_{2,n+2}(T_a, f, -1)\right]. \end{aligned} \tag{12}$$

Each of (10)–(12) can be verified by simply checking that the polynomial on the right-hand side satisfies the defining conditions (in terms of degree and values at the interpolation nodes (3)) of the polynomial on the left-hand side. Also note (Byrne *et al.* [4, Section 1]) that $H_{2,n}(T, f)$ has the explicit formula

$$H_{2,n}(T, f, x) = \tfrac{1}{n^3}\,(T_n(x))^3\,S_n(f, x) \tag{13}$$

where, with $x_k = x_{k,n+2}$ defined by (3),

$$S_n(f, x) = \sum_{k=1}^{n} (-1)^{k-1}\sqrt{1 - x_k^2}\left(\tfrac{1 - x x_k}{(x - x_k)^3} - \tfrac{x_k}{2(x - x_k)^2} + \tfrac{n^2 - 1}{2(x - x_k)}\right) f(x_k). \tag{14}$$

The results of the following two lemmas will be used to develop alternative representations for $R_{2,n+2}(T_a, f)$ and $H_{2,n+2}(T_a, f)$ to those in (11) and (12).

LEMMA 4. *For* $x_k = x_{k,n+2}$ *defined by* (3) *and* $f \in C[-1, 1]$, *the following summation formulas hold.*

$$\tfrac{T_n(x)}{n} \sum_{k=1}^{n} \tfrac{(-1)^{k-1}\sqrt{1 - x_k^2}}{x - x_k} f(x_k) = L_n(T, f, x) \tag{15}$$

$$\frac{T_n(x)}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(x-x_k)^2}f(x_k)=\frac{T_n'(x)}{T_n(x)}L_n(T,f,x)-L_n'(T,f,x) \tag{16}$$

$$\begin{aligned}\frac{T_n(x)}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(x-x_k)^3}f(x_k)&=\left[\left(\frac{T_n'(x)}{T_n(x)}\right)^2-\frac{1}{2}\frac{T_n''(x)}{T_n(x)}\right]L_n(T,f,x)\\&\quad-\frac{T_n'(x)}{T_n(x)}L_n'(T,f,x)+\frac{1}{2}L_n''(T,f,x)\end{aligned} \tag{17}$$

$$\begin{aligned}&\frac{T_n(x)}{n}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(x-x_k)^4}f(x_k)=\\&=\left[\frac{1}{6}\frac{T_n'''(x)}{T_n(x)}-\frac{T_n'(x)T_n''(x)}{(T_n(x))^2}+\left(\frac{T_n'(x)}{T_n(x)}\right)^3\right]L_n(T,f,x)\\&\quad+\left[\frac{1}{2}\frac{T_n''(x)}{T_n(x)}-\left(\frac{T_n'(x)}{T_n(x)}\right)^2\right]L_n'(T,f,x)+\frac{1}{2}\frac{T_n'(x)}{T_n(x)}L_n''(T,f,x)-\frac{1}{6}L_n'''(T,f,x)\end{aligned} \tag{18}$$

Proof. The formula (15) for Lagrange interpolation on the Chebyshev nodes is well-known (see, for example, Rivlin [12, Section 1.3]). The remaining formulas (16)–(18) follow by successive differentiation of (15). ☐

LEMMA 5. *For $x_k = x_{k,n+2}$ defined by* (3)*, the following summation formulas hold.*

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{1-x_k}=n \tag{19}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^2}=n^3 \tag{20}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^3}=\frac{1}{6}n^3(5n^2+1) \tag{21}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^4}=\frac{1}{90}n^3(61n^4+25n^2+4) \tag{22}$$

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1+x_k)^r}=(-1)^{n+1}\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^r},\quad r=1,2,3,\ldots \tag{23}$$

Proof. If $f(x)\equiv 1$, then $L_n(T,f,x)\equiv 1$, so (15) gives

$$\sum_{k=1}^{n}\frac{(-1)^{k-1}\sqrt{1-x_k^2}}{x-x_k}=\frac{n}{T_n(x)}.$$

Putting $x = 1$ gives (19). The formulas (20)–(22) follow from (16)–(18) in similar fashion after noting that if $f(x)\equiv 1$, then $L_n^{(r)}(T,f,x)\equiv 0$ for $r\geq 1$, and (Rivlin [12, p. 38])

$$T_n^{(r)}(1)=\frac{n^2(n^2-1^2)(n^2-2^2)\ldots(n^2-(r-1)^2)}{1\,.\,3\,.\,5\ldots(2r-1)}. \tag{24}$$

Finally, (23) is a straightforward consequence of the symmetry of the Chebyshev nodes (i.e. $x_{k,n+2} = -x_{n-k+1,n+2}$ for $1 \le k \le n$). $\square$

We now obtain alternative representations for the interpolation polynomials $R_{2,n+2}(T_a, f)$ and $H_{2,n+2}(T_a, f)$ to those in (11) and (12).

LEMMA 6. *For $x_k = x_{k,n+2}$ defined by* (3) *and $f \in C[-1,1]$,*

$$\begin{aligned} R_{2,n+2}(T_a,f,x) =& H_{2,n}(T,f,x) \\ &+ \tfrac{1}{4}(T_n(x))^3 \Big\{ (1+x)^2(2-x)\,[f(1)-H_{2,n}(T,f,1)] \\ &+ (-1)^n(1-x)^2(2+x)\,[f(-1)-H_{2,n}(T,f,-1)] \\ &+ (1-x^2)\,[(1+x)A_{1,n}-(1-x)B_{1,n}] \Big\} \end{aligned} \tag{25}$$

and

$$\begin{aligned} H_{2,n+2}(T_a,f,x) =& R_{2,n+2}(T_a,f,x) + \tfrac{1}{16}(T_n(x))^3(1-x^2)^2 \times \\ &\times \Big[3x\{[f(1) - H_{2,n}(T,f,1)] - (-1)^n\,[f(-1) - H_{2,n}(T,f,-1)]\} \\ &+ (1+x)(A_{2,n}+B_{1,n}) - (1-x)(B_{2,n}+A_{1,n}) \Big] \end{aligned} \tag{26}$$

where

$$A_{1,n} = \tfrac{1}{n^3} \sum_{k=1}^{n} (-1)^{k-1} \sqrt{1-x_k^2} \left(\tfrac{3}{(1-x_k)^2} + \tfrac{n^2-1}{2(1-x_k)} \right) \tfrac{f(1)-f(x_k)}{1-x_k}, \tag{27}$$

$$B_{1,n} = \tfrac{1}{n^3} \sum_{k=1}^{n} (-1)^{k-1} \sqrt{1-x_k^2} \left(\tfrac{3}{(1+x_k)^2} + \tfrac{n^2-1}{2(1+x_k)} \right) \tfrac{f(-1)-f(x_k)}{1+x_k}, \tag{28}$$

$$A_{2,n} = \tfrac{1}{n^3} \sum_{k=1}^{n} (-1)^{k-1} \sqrt{1-x_k^2} \left(\tfrac{15}{(1-x_k)^3} + \tfrac{n^2+2}{(1-x_k)^2} + \tfrac{n^2-1}{1-x_k} \right) \tfrac{f(1)-f(x_k)}{1-x_k}, \tag{29}$$

$$B_{2,n} = \tfrac{1}{n^3} \sum_{k=1}^{n} (-1)^{k-1} \sqrt{1-x_k^2} \left(\tfrac{15}{(1+x_k)^3} + \tfrac{n^2+2}{(1+x_k)^2} + \tfrac{n^2-1}{1+x_k} \right) \tfrac{f(-1)-f(x_k)}{1+x_k}. \tag{30}$$

Proof. For convenience, denote the RHS of (25) by $r(x)$. To establish (25) it is sufficient to show $r(x)$ satisfies the $3n+4$ defining conditions (5) of $R_{2,n+2}(T_a, f)$. In fact, apart from $r'(1) = r'(-1) = 0$, the conditions are verified easily. To show $r'(1) = 0$, firstly observe from (5) that (upon using $T_n'(1) = n^2$)

$$r'(1) = H_{2,n}'(T,f,1) + 3n^2\left(f(1) - H_{2,n}(T,f,1)\right) - A_{1,n}.$$

However, by (13) and (14),

$$H_{2,n}'(T,f,1) = 3n^2 H_{2,n}(T,f,1) + \tfrac{1}{n^3} S_n'(f,1),$$

where

$$S_n'(f,1) = -\sum_{k=1}^{n}(-1)^{k-1}\sqrt{1-x_k^2}\left(\tfrac{3}{(1-x_k)^3}+\tfrac{n^2-1}{2(1-x_k)^2}\right)f(x_k). \tag{31}$$

Thus

$$r'(1) = 3n^2 f(1) + \tfrac{1}{n^3}S_n'(f,1) - A_{1,n},$$

and this is zero by (20), (21) and (27). The result $r'(-1) = 0$ is established by similar means.

Again for convenience, denote the RHS of (26) by $h(x)$. By the same arguments as above, it is evident that (26) will be proved if it can be shown that $h''(1) = h''(-1) = 0$. To show $h''(1) = 0$, begin by noting that from (25) and (26),

$$h''(1) = H_{2,n}''(T,f,1) + (7n^4 - n^2)\left(f(1) - H_{2,n}(T,f,1)\right) - (6n^2+2)A_{1,n} + A_{2,n}. \tag{32}$$

However, by (13) and (14),

$$H_{2,n}''(T,f,1) = (7n^4 - n^2)H_{2,n}(T,f,1) + \tfrac{6}{n}S_n'(f,1) + \tfrac{1}{n^3}S_n''(f,1), \tag{33}$$

where

$$S_n''(f,1) = \sum_{k=1}^{n}(-1)^{k-1}\sqrt{1-x_k^2}\left(\tfrac{3x_k+12}{(1-x_k)^4}+\tfrac{n^2-1}{(1-x_k)^3}\right)f(x_k). \tag{34}$$

By substituting (27), (29), (31), (33) and (34) into (32), and employing summation formulas (20)–(22), it follows (after somewhat tedious calculations) that $h''(1) = 0$. The result $h''(-1) = 0$ is proved in similar fashion. $\square$

The next result characterizes the quantities $A_{i,n}$ and $B_{i,n}$ of Lemma 6 in terms of values of Lagrange interpolation polynomials and their derivatives.

LEMMA 7. *For $i = 1,2$, the quantities $A_{i,n}$ and $B_{i,n}$ that are defined by* (27)–(30) *can be written as*

$$A_{1,n} = 3n^2\left[f(1) - L_n(T,f,1)\right] + \tfrac{7n^2-1}{2n^2}L_n'(T,f,1) - \tfrac{3}{2n^2}L_n''(T,f,1), \tag{35}$$

$$\begin{aligned} B_{1,n} = &(-1)^{n+1}\big[3n^2\left[f(-1) - L_n(T,f,-1)\right] \\ &- \tfrac{7n^2-1}{2n^2}L_n'(T,f,-1) - \tfrac{3}{2n^2}L_n''(T,f,-1)\big], \end{aligned} \tag{36}$$

$$\begin{aligned} A_{2,n} = &(11n^4+7n^2)\left[f(1) - L_n(T,f,1)\right] + \tfrac{27n^4+11n^2-2}{2n^2}L_n'(T,f,1) \\ &- \tfrac{8n^2+1}{n^2}L_n''(T,f,1) + \tfrac{5}{2n^2}L_n'''(T,f,1), \end{aligned} \tag{37}$$

$$\begin{aligned} B_{2,n} = &(-1)^{n+1}\big[(11n^4+7n^2)\left[f(-1) - L_n(T,f,-1)\right] \\ &- \tfrac{27n^4+11n^2-2}{2n^2}L_n'(T,f,-1) - \tfrac{8n^2+1}{n^2}L_n''(T,f,-1) - \tfrac{5}{2n^2}L_n'''(T,f,-1)\big]. \end{aligned} \tag{38}$$

Proof. To obtain (35), begin by writing (27) as

$$A_{1,n} = \tfrac{3}{n^3}\sum_{k=1}^{n} \tfrac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^3} g(x_k) + \tfrac{n^2-1}{2n^3}\sum_{k=1}^{n} \tfrac{(-1)^{k-1}\sqrt{1-x_k^2}}{(1-x_k)^2} g(x_k),$$

where $g(x) = f(1)-f(x)$. Now apply the identities (16) and (17) at $x = 1$, noting that $L_n(T, g, x) = f(1) - L_n(T, f, x)$ and using the result (24) for $T_n^{(r)}(1)$. To obtain (36), apply (16) and (17) at $x = -1$ to (28) with $g(x) = f(-1)-f(x)$, and use $T_n^{(r)}(-1) = (-1)^{n+r}T_n^{(r)}(1)$. The remaining expressions (37) and (38) are proved in a similar fashion. □

For a function f that has continuous derivatives on $[-1, 1]$, the following lemma characterizes the limiting behaviour of the $A_{i,n}$ and $B_{i,n}$ as $n \to \infty$ in terms of these derivatives. The proof of the lemma uses the following result which is a special case of a more general theorem concerning Lagrange interpolation on Jacobi nodes that is due to Neckermann and Runck [10, Satz 2, p. 168]:

(39) If $m > 0$ and $f^{(2m)} \in C[-1, 1]$, then $\lim_{n\to\infty} \|L_n^{(m)}(T, f) - f^{(m)}\| = 0.$

Lemma 8. *Suppose $A_{i,n}$ and $B_{i,n}$ are defined by* (27)*–*(30). *If $f \in C^2[-1, 1]$, then*

$$\lim_{n\to\infty} A_{1,n} = \tfrac{7}{2} f'(1), \tag{40}$$

$$\lim_{n\to\infty} (-1)^n B_{1,n} = \tfrac{7}{2} f'(-1). \tag{41}$$

If $f \in C^4[-1, 1]$, then

$$\lim_{n\to\infty} \Big(A_{2,n} - \tfrac{27n^2}{2} f'(1)\Big) = \tfrac{11}{2} f'(1) - 8f''(1), \tag{42}$$

$$\lim_{n\to\infty} \Big((-1)^n B_{2,n} - \tfrac{27n^2}{2} f'(-1)\Big) = \tfrac{11}{2} f'(-1) + 8f''(-1). \tag{43}$$

Proof. Suppose $f \in C^2[-1, 1]$. We work with the expression (35) for $A_{1,n}$. Firstly, with $x_1 = \cos(\pi/(2n))$, it follows from the Mean Value Theorem that there exists $\alpha \in (x_1, 1)$ so that

$$\begin{aligned}\Big|f(1) - L_n(T, f, 1)\Big| &= \Big|\, (f(1) - L_n(T, f, 1)) - (f(x_1) - L_n(T, f, x_1))\,\Big| \\ &= \Big|(1 - x_1)\,(f'(\alpha) - L_n'(T, f, \alpha))\,\Big| \\ &\le 2\sin^2(\tfrac{\pi}{4n})\, \|f' - L_n'(T, f)\|.\end{aligned}$$

Thus, by (39),

$$\lim_{n\to\infty} n^2\, [f(1) - L_n(T, f, 1)] = 0. \tag{44}$$

Also, if $\mathcal{P}_n$ denotes the set of polynomials of degree at most n, let q be the best uniform approximation to f' in $\mathcal{P}_{n-3}$. Then, by Markov's inequality,

$$\begin{aligned}|L_n''(T,f,1)| &\le \|L_n''(T,f)-q'\|+\|q'\|\\ &\le (n-2)^2\|L_n'(T,f)-q\|+\|q'\|\\ &\le (n-2)^2\left(\|L_n'(T,f)-f'\|+\|f'-q\|\right)+\|q'\|.\end{aligned}$$

Now, there exists an absolute constant c so that $\|q'\| \le c(n-3)^2\omega(\frac{1}{n-3};f')$ (see Szabados and Vértesi [14, p. 284]), so

$$\tfrac{1}{n^2}|L_n''(T,f,1)| \le \|L_n'(T,f)-f'\|+\|f'-q\|+c\,\omega(\tfrac{1}{n-3};f').$$

Since $f \in C^2[-1,1]$, it follows from (39) and the Weierstrass approximation theorem that

$$\lim_{n\to\infty}\tfrac{1}{n^2}L_n''(T,f,1)=0. \tag{45}$$

Substituting (44) and (45) into (35) and using (39) then gives the result (40). The result (41) is established from (36) in near-identical fashion.

Now suppose $f \in C^4[-1,1]$. Here we work with the expression (37) for $A_{2,n}$. With $x_k = \cos((2k-1)\pi/(2n))$, by the Mean Value Theorem there exists $\beta \in (x_2,x_1)$ so that $f'(\beta)-L_n'(T,f,\beta)=0$. Thus, with $\alpha \in (x_1,1)$ as above, there exists $\gamma \in (\beta,\alpha)$ such that

$$\begin{aligned}|f(1)-L_n(T,f,1)| =&\left|(1-x_1)\left(f'(\alpha)-L_n'(T,f,\alpha)\right)\right|\\ =&\Big|(1-x_1)\left[\left(f'(\alpha)-L_n'(T,f,\alpha)\right)-\left(f'(\beta)-L_n'(T,f,\beta)\right)\right]\Big|\\ =&\left|(1-x_1)(\alpha-\beta)\left(f''(\gamma)-L_n''(T,f,\gamma)\right)\right|\\ \le&4\sin^2(\tfrac{\pi}{4n})\sin^2\left(\tfrac{3\pi}{4n}\right)\|f''-L_n''(T,f)\|,\end{aligned}$$

and so, by (39),

$$\lim_{n\to\infty} n^4\left[f(1)-L_n(T,f,1)\right]=0. \tag{46}$$

Similarly,

$$\begin{aligned}|f'(1)-L_n'(T,f,1)| &= \left|\left(f'(1)-L_n'(T,f,1)\right)-\left(f'(\beta)-L_n'(T,f,\beta)\right)\right|\\ &\le 2\sin^2\left(\tfrac{3\pi}{4n}\right)\|f''-L_n''(T,f)\|,\end{aligned}$$

so

$$\lim_{n\to\infty} n^2\left[f'(1)-L_n'(T,f,1)\right]=0. \tag{47}$$

Now let r be the best uniform approximation to f'' in $\mathcal{P}_{n-4}$. By the method used above to derive (45) it follows that

$$\tfrac{1}{n^2}|L_n'''(T,f,1)| \le \|L_n''(T,f)-f''\|+\|f''-r\|+c\,\omega(\tfrac{1}{n-4};f''),$$

and so

$$\lim_{n\to\infty}\tfrac{1}{n^2}L_n'''(T,f,1)=0. \tag{48}$$

Substituting (46)–(48) into (37) and using (39) then gives (42). The result (43) is established similarly. □

We have now developed all the preliminary results needed to establish our theorem.

Proof of Theorem 2. Firstly observe that a) is an immediate consequence of the representation (10) for $Q_{2,n+2}(T_a, f)$.

b) Suppose $f \in C^2[-1,1]$ and $x \in (-1,1)$. Then $H_{2,n}(T,f)$ converges uniformly to f, so by (25), $R_{2,n+2}(T_a,f,x)$ converges (pointwise or uniformly) to $f(x)$ if and only if

$$\lim_{n\to\infty}(T_n(x))^3\left[(1+x)A_{1,n}-(1-x)B_{1,n}\right]=0,$$

which (by (40) and (41)) is equivalent to

$$\lim_{n\to\infty}(T_n(x))^3\Big[(1+x)f'(1)+(-1)^{n+1}(1-x)f'(-1)\Big]=0. \tag{49}$$

Clearly, then, $R_{2,n+2}(T_a,f)$ converges uniformly to f if $f'(1)=f'(-1)=0$.

On the other hand, since $T_n(0)=0$ for n odd and $|T_n(0)|=1$ for n even, (49) holds true at $x=0$ if and only if $f'(1)=f'(-1)$. Thus $R_{2,n+2}(T_a,f,0)\to f(0)$ if and only if $f'(1)=f'(-1)$.

For fixed x with $0<|x|<1$, it is well-known that there exists a subsequence $\{n_r\}_{r=1}^{\infty}$ of natural numbers so that $|T_{n_r}(x)|\to 1$. Since $|T_{n_r+1}(x)|\to|x|$, it follows that there is a subsequence $\{m_r\}_{r=1}^{\infty}$ of natural numbers that contains infinitely many odd values and infinitely many even values and satisfies $|T_{m_r}(x)|\ge|x|/2$ for all r. Thus (49) holds true if and only if $(1+x)f'(1)+(1-x)f'(-1)=0$ and $(1+x)f'(1)-(1-x)f'(-1)=0$, which is equivalent to $f'(1)=f'(-1)=0$.

c) Suppose $f\in C^4[-1,1]$ and $x\in(-1,1)$. Again $H_{2,n}(T,f)$ converges uniformly to f, so by (25) and (26),

$$\begin{aligned}&H_{2,n+2}(T_a,f,x)=\\&=f(x)+\tfrac{1}{16}(T_n(x))^3(1-x^2)\big[(1+x)^2(3-x)A_{1,n}\\&\quad-(1-x)^2(3+x)B_{1,n}+(1+x)^2(1-x)A_{2,n}-(1-x)^2(1+x)B_{2,n}\big]+o(1)\end{aligned}$$

where, here and subsequently, the $o(1)$ term is uniform in x. Then by (40)-(43),

$$\begin{aligned}&H_{2,n+2}(T_a,f,x)=\\&=f(x)+\tfrac{1}{32}(T_n(x))^3(1-x^2)\Big[7(1+x)^2(3-x)f'(1)\\&\quad-7(-1)^n(1-x)^2(3+x)f'(-1)+(1+x)^2(1-x)\big[(27n^2+11)f'(1)-16f''(1)\big]\\&\quad-(-1)^n(1-x)^2(1+x)\big[(27n^2+11)f'(-1)+16f''(-1)\big]\Big]+o(1).\end{aligned} \tag{50}$$

It is obvious from this expression that $H_{2,n+2}(T_a,f)$ converges uniformly to f if $f'(1)=f'(-1)=f''(1)=f''(-1)=0$.

On the other hand, (50) shows that $H_{2,n+2}(T_a, f, 0) \to f(0)$ if and only if

$$\lim_{\substack{n\to\infty \\ n \text{ even}}} (27n^2 + 32)\big[f'(1) - f'(-1)\big] - 16\big[f''(1) + f''(-1)\big] = 0,$$

which is equivalent to $f'(1) = f'(-1)$ and $f''(1) = -f''(-1)$. Furthermore, if $f'(1) \neq f'(-1)$, (9) follows from (50).

For fixed x with $0 < |x| < 1$, suppose $H_{2,n+2}(T_a, f, x) \to f(x)$. Then, with $\{m_r\}$ defined as in (b) above, it follows from (50) that

$$\begin{aligned}(51) \quad \lim_{r\to\infty} \Big[&7(1+x)^2(3-x)f'(1) - 7(-1)^{m_r}(1-x)^2(3+x)f'(-1) \\ &+ (1+x)^2(1-x)\big[(27m_r^2 + 11)f'(1) - 16f''(1)\big] \\ &- (-1)^{m_r}(1-x)^2(1+x)\big[(27m_r^2 + 11)f'(-1) + 16f''(-1)\big]\Big] = 0.\end{aligned}$$

Because $\{m_r\}$ has infinitely many values of each parity, (51) implies that $(1+x)f'(1) + (1-x)f'(-1) = 0$ and $(1+x)f'(1) - (1-x)f'(-1) = 0$, which means $f'(1) = f'(-1) = 0$. A similar argument then shows that $f''(1) = f''(-1) = 0$. Finally, if $f'(1)$ and $f'(-1)$ are not both 0, (8) is a consequence of (50) and the definition of m_r. □

Remark. Theorem 1 for $(0, 1)$ HFI does not place any conditions on the derivatives of f, other than the existence of f' at ± 1. By contrast, our Theorem 2 for $(0, 1, 2)$ HFI imposes quite stringent conditions on the derivatives of f. However, it seems difficult to fully extend Bojanic's methods to $(0, 1, 2)$ HFI, essentially because the positivity of terms that occur when working with $(0, 1)$ HFI no longer applies to the corresponding terms in $(0, 1, 2)$ HFI. Nevertheless, it would be of interest to know whether the conditions of our theorem can be weakened.

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