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APPROXIMATION OF THE HILBERT TRANSFORM IN THE LEBESGUE SPACES

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Abstract. The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform is a multiplier operator and is widely used in the theory of Fourier transforms. The Hilbert transform is the main part of the singular integral equations on the real line. Therefore, approximations of the Hilbert transform are of great interest. Many papers have dealt with the numerical approximation of the singular integrals in the case of bounded intervals. On the other hand, the literature concerning the numerical integration on unbounded intervals is by far poorer than the one on bounded intervals. The case of the Hilbert Transform has been considered very little. This article is devoted to the approximation of the Hilbert transform in Lebesgue spaces by operators which introduced by V.R. Kress and E. Mortensen to approximate the Hilbert transform of analytic functions in a strip. In this paper, we prove that the approximating operators are bounded maps in Lebesgue spaces and strongly converges to the Hilbert transform in these spaces.

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Keywords. Hilbert transform, singular integral, approximation, Lebesgue space.

1. INTRODUCTION

The Hilbert transform of a function $u \in L_p(\mathbb{R})$, $1 \le p < \infty$ is defined as the Cauchy principle value integral [18]

$$(Hu)(t) = \frac{1}{\pi} \int_R \frac{u(\tau)}{t-\tau} d\tau, \quad t \in \mathbb{R},$$

where the integral is understood in the Cauchy principal value sense. It is well known (see [14, 18, 32]) that the Hilbert transform of the function $u \in L_p(\mathbb{R})$, $1 \le p < \infty$, exists for almost all values of $t \in \mathbb{R}$. In case $1 < p < \infty$,

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the Hilbert transform is a bounded map in the space $L_p(\mathbb{R})$ and satisfies the equation:

$$H^2 = -I.$$

The Hilbert transform plays an important role in the theory and practice of signal processing operations in continuous system theory because of its relevance to such problems as envelope detection and demodulation, as well as its use in relating the real and imaginary components, and the magnitude and phase components of spectra. The Hilbert transform is a multiplier operator and is widely used in the theory of Fourier transforms. The Hilbert transform is the main part of the singular integral equations on the real line (see [24]). Therefore, approximations of the Hilbert transform are of great interest.

Many papers have dealt with the numerical approximation of the Hilbert Transform in the case of bounded intervals and the reader can refer to [1, 3, 6, 7, 9, 10, 12, 13, 15, 16, 17, 20, 21, 22, 25, 28, 30, 31, 32, 37, 38] and the references given there. On the other hand, the literature concerning the numerical integration on unbounded intervals is by far poorer than the one on bounded intervals. The case of the Hilbert Transform has been considered very little and the reader can consult [2, 8, 11, 12, 19, 20, 23, 26, 27, 34, 35, 36, 39]. In particular, in [19] the authors assume that the function u is analytic in the strip $\{z \in \mathbb{C} : |\Im z| < d\}$, in which case they show that the series $\frac{2}{\pi}\sum_{k\in\mathbb{Z},k\neq even} \frac{u(t+k\delta)}{-k}$ uniformly converges to $(Hu)(t)$ as $\delta \to 0$. In [5] the author replaces the above series with the following one $\frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+(k+1/2)\delta)}{-k-1/2}$ for a suitable choice of the step $\delta \to 0$.

This article is devoted to the approximation of the Hilbert transform of functions from $L_p(\mathbb{R})$ by operators of the form

$$(H_\delta u)(t) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+(k+1/2)\delta)}{-k-1/2}, \quad \delta > 0$$

which were introduced in [19].

In Section 2 we present the properties of the approximating operators H_δ. We show that the operators H_δ are bounded maps in the space $L_p(\mathbb{R})$, $1 < p < \infty$ and

$$H_\delta^2 = -I$$

in $L_p(\mathbb{R})$ (Theorem 2).

In Section 3 we give an approximation of the singular integral with Hilbert kernel

$$(S\varphi)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cot\frac{t-\tau}{2}\varphi(\tau)d\tau, \quad t \in T = [-\pi, \pi)$$

by a sequence of operators

$$(S_n\varphi)(t) = \frac{1}{n}\sum_{k=0}^{n-1} \cot\left(-\frac{\pi(2k+1)}{2n}\right)\varphi\left(t + \frac{\pi(2k+1)}{n}\right), \quad n \in \mathbb{N}.$$

in $L_p(T)$. We show that the operators S_n are uniformly bounded in $L_p(T)$ and strongly converges to the operator S in $L_p(T)$, $1 < p < \infty$ (Theorems 3 and 4).

In Section 4 we give an approximation of the Hilbert transform H by the operators H_δ. We show that for any $\delta > 0$ the sequence of operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_p(\mathbb{R})$, $1 < p < \infty$ (Theorem 9).

Note that in this paper the singular integral with Hilbert kernel and the Hilbert transform is approximated by operators preserving the main properties of these operators (see: Theorem 2 and (6), (7)). This leads to give an approximation of the singular integral and the Hilbert transform of the functions from L_p, $1 < p < \infty$, but other approximate methods can only be applied to continuous or piecewise continuous functions.

2. PROPERTIES OF THE APPROXIMATING OPERATORS H_δ

Let l_p, $1 \le p < \infty$, the space of all sequences $b = \{b_n\}_{n\in\mathbb{Z}}$ with finite norm $\|b\|_{l_p} = (\sum_{n\in\mathbb{Z}} |b_n|^p)^{1/p}$. The sequence $h(b) = \{(h(b))_n\}_{n\in\mathbb{Z}}$ is called the discrete Hilbert transform of the sequence $b = \{b_n\}_{n\in\mathbb{Z}}$, where $(h(b))_n = \sum_{m\neq n} \frac{b_m}{n-m}$, $n \in \mathbb{Z}$.

M. Riesz (see [29]) proved that if $b \in l_p$, $1 < p < \infty$, then $h(b) \in l_p$ and the inequality

$$\|h(b)\|_{l_p} \le C_p \|b\|_{l_p} \tag{1}$$

holds, where C_p is constant depending only on p.

We will use a modified version of the discrete Hilbert transform: $(\tilde{h}(b))_n = \sum_{m\in\mathbb{Z}} \frac{b_m}{n-m-1/2}$, $n \in \mathbb{Z}$. K. Andersen [4] proved that the inequality (1) is also valid for the transform $\tilde{h}$, that is, there exist $\tilde{C}_p > 0$ such that the inequality

$$\|\tilde{h}(b)\|_{l_p} \le \tilde{C}_p \|b\|_{l_p} \tag{2}$$

holds for any $b \in l_p$, $1 < p < \infty$.

In the following theorems we prove that the operators H_δ are bounded maps in the space $L_p(\mathbb{R})$ and $H_\delta^2 = -I$ in $L_p(\mathbb{R})$, $1 < p < \infty$.

THEOREM 1. *For any $\delta > 0$ the operator H_δ is bounded in the space $L_p(\mathbb{R})$, $1 < p < \infty$, and the inequality*

$$\|H_\delta\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} \le \|\tilde{h}\|_{l_p\to l_p} \tag{3}$$

holds.

Proof. Let $u \in L_p(\mathbb{R})$, $1 < p < \infty$. For any $t \in \mathbb{R}$

$$\tilde{h}\Big(\{u(t+\delta/2+n\delta)\}_{n\in\mathbb{Z}}\Big) = \left\{\frac{1}{\pi}\sum_{m\in\mathbb{Z}} \frac{u(t+\delta/2+m\delta)}{n-m-1/2}\right\}_{n\in\mathbb{Z}}$$

$$= \left\{\frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+\delta/2+k\delta+n\delta)}{-k-1/2}\right\}_{n\in\mathbb{Z}} = \{(H_\delta u)(t+n\delta)\}_{n\in\mathbb{Z}}.$$

Then, by inequality (2), for almost all $t \in \mathbb{R}$

$$\|\{(H_\delta u)(t+n\delta)\}_{n\in\mathbb{Z}}\|_{l_p} = \|\tilde{h}(\{u(t+\delta/2+n\delta)\}_{n\in\mathbb{Z}})\|_{l_p} \\ \le \|\tilde{h}\|_{l_p\to l_p} \cdot \left\|\{u(t+\delta/2+n\delta)\}_{n\in\mathbb{Z}}\right\|_{l_p}.$$

It follows that

$$\begin{aligned}\|H_\delta u\|^p_{L_p(\mathbb{R})} &= \int_R |(H_\delta u)(t)|^p dt = \sum_{n\in\mathbb{Z}} \int_{(n-1/2)\delta}^{(n+1/2)\delta} |(H_\delta u)(t)|^p dt \\ &= \sum_{n\in\mathbb{Z}} \int_{-\delta/2}^{\delta/2} |(H_\delta u)(t+n\delta)|^p dt = \int_{-\delta/2}^{\delta/2} \sum_{n\in\mathbb{Z}} |(H_\delta u)(t+n\delta)|^p dt \\ &= \int_{-\delta/2}^{\delta/2} \|\{(H_\delta u)(t+n\delta)\}_{n\in\mathbb{Z}}\|^p_{l_p} dt \\ &\le \|\tilde{h}\|^p_{l_p\to l_p} \cdot \int_{-\delta/2}^{\delta/2} \|\{u(t+\delta/2+n\delta)\}_{n\in\mathbb{Z}}\|^p_{l_p} dt \\ &= \|\tilde{h}\|^p_{l_p\to l_p} \cdot \int_{-\delta/2}^{\delta/2} \sum_{n\in\mathbb{Z}} |u(t+\delta/2+n\delta)|^p dt \\ &= \|\tilde{h}\|^p_{l_p\to l_p} \cdot \sum_{n\in\mathbb{Z}} \int_{-\delta/2}^{\delta/2} |u(t+\delta/2+n\delta)|^p dt \\ &= \|\tilde{h}\|^p_{l_p\to l_p} \cdot \sum_{n\in\mathbb{Z}} \int_{n\delta}^{(n+1)\delta} |u(t)|^p dt = \|\tilde{h}\|^p_{l_p\to l_p} \cdot \|u\|^p_{L_p(\mathbb{R})}.\end{aligned}$$

□

THEOREM 2. *For any $\delta > 0$ and $u \in L_p(\mathbb{R})$, $1 < p < \infty$ the following inequality holds:*

$$H_\delta(H_\delta u)(t) = -u(t). \tag{4}$$

Proof. For any $u \in L_p(\mathbb{R})$ we have

$$\begin{aligned}H_\delta(H_\delta u)(t) &= -\tfrac{1}{\pi} \sum_{k\in\mathbb{Z}} \tfrac{(H_\delta u)(t+(k+1/2)\delta)}{k+1/2} = \tfrac{1}{\pi} \sum_{k\in\mathbb{Z}} \tfrac{1}{k+1/2} \cdot \tfrac{1}{\pi} \sum_{m\in\mathbb{Z}} \tfrac{u(t+(k+m+1)\delta)}{m+1/2} \\ &= \tfrac{1}{\pi^2} \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \tfrac{u(t+(k+m+1)\delta)}{(k+1/2)(m+1/2)} = \tfrac{1}{\pi^2} \sum_{k\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \tfrac{u(t+n\delta)}{(k+1/2)(n-k-1/2)} \\ &= \tfrac{1}{\pi^2} \sum_{n\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} \tfrac{1}{(k+1/2)(n-k-1/2)} \right) u(t+n\delta). \qquad (5)\end{aligned}$$

Since for $n = 0$

$$\sum_{k\in\mathbb{Z}} \tfrac{1}{(k+1/2)(n-k-1/2)} = -4 \sum_{k\in\mathbb{Z}} \tfrac{1}{(2k+1)^2} = -\pi^2,$$

and for $n \neq 0$

$$\sum_{k\in\mathbb{Z}} \tfrac{1}{(k+1/2)(n-k-1/2)} = \sum_{k\in\mathbb{Z}} \tfrac{1}{n}\left[\tfrac{1}{k+1/2} + \tfrac{1}{n-k-1/2}\right]$$
$$= \tfrac{1}{n} \lim_{N\to\infty} \sum_{|k|\le N} \left[\tfrac{1}{k+1/2} + \tfrac{1}{n-k-1/2}\right] = 0,$$

then equality (4) follows from (5). □

3. APPROXIMATION OF THE SINGULAR INTEGRAL WITH HILBERT KERNEL

Denote by $L_p(T)$, $1 \le p < \infty$, the space of all measurable, 2π-periodic functions with finite norm $\|\varphi\|_{L_p(T)} = (\int_T |\varphi(t)|^p dt)^{1/p}$, where $T = [-\pi, \pi)$, and by $L_p([a,b])$ the space of all measurable functions on the interval $[a,b] \subset \mathbb{R}$ with finite norm $\|\varphi\|_{L_p([a,b])} = \left(\int_a^b |\varphi(t)|^p dt\right)^{1/p}$.

It is well known that (see [40]) the singular integral with Hilbert kernel

$$(S\varphi)(t) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \cot \tfrac{t-\tau}{2} \varphi(\tau) d\tau, \quad t \in T,$$

is a bounded map in the space $L_p(T)$, $1 < p < \infty$ and for any $\varphi \in L_p(T)$

$$(S^2\varphi)(t) = -\varphi(t) + \tfrac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(\tau) d\tau, \quad t \in T.$$

Consider in $L_p(T)$, $1 < p < \infty$ the sequence of operators

$$(S_n\varphi)(t) = \tfrac{1}{n} \sum_{k=0}^{n-1} \cot\left(-\tfrac{\pi(2k+1)}{2n}\right) \varphi\left(t + \tfrac{\pi(2k+1)}{n}\right), \quad n \in \mathbb{N}.$$

It is easy to verify that if

$$\varphi(t) = \tfrac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mt + b_m \sin mt),$$

then

$$(S_n\varphi)(t) = \sum_{m=1}^{\infty} \lambda_m^{(n)} (a_m \cos mt + b_m \sin mt),$$

where $\lambda_m^{(n)} = 1$ for $m = \overline{1, n-1}$, $\lambda_n^{(n)} = \lambda_{2n}^{(n)} = 0$, $\lambda_m^{(n)} = -1$ for $m = \overline{n+1, 2n-1}$ and $\lambda_{m+2n}^{(n)} = \lambda_m^{(n)}$ for $m \in \mathbb{Z}$. It follows from here that for any trigonometric polynomial $P(t)$ of order at most $n-1$

$$(S_nP)(t) = (SP)(t), \tag{6}$$

and for any $\varphi \in L_p(T)$

$$(S_n^2\varphi)(t) = -\varphi(t) + \tfrac{1}{n} \sum_{k=0}^{n-1} \varphi(t + \tfrac{2\pi k}{n}), \quad n \in \mathbb{N}. \tag{7}$$

In the following theorems we prove that the sequence of operators S_n are uniformly bounded in $L_p(T)$ and strongly converges to the operator S in $L_p(T)$, $1 < p < \infty$.

THEOREM 3. *Operators S_n are uniformly bounded in $L_p(T)$, $1 < p < \infty$, and for any $n \in \mathbb{N}$ the inequality*

$$\|S_n\|_{L_p(T)\to L_p(T)} \le 4 + 2\|\tilde{h}\|_{l_p \to lp}$$

holds.

Proof. Let $\varphi \in L_p(T)$. Define the function $u(t) = \varphi(t)$ for $t \in [-2\pi, 2\pi]$ and $u(t) = 0$ for $t \in \mathbb{R} \setminus [-2\pi, 2\pi]$. Then $u \in L_p(\mathbb{R})$, and therefore, it follows from Theorem 1 that for any $\delta > 0$

$$\|H_\delta u\|_{L_p(\mathbb{R})} \le \|\tilde{h}\|_{l_p \to l_p} \cdot \|u\|_{L_p(\mathbb{R})} = 2\|\tilde{h}\|_{l_p \to l_p} \cdot \|\varphi\|_{L_p(T)}. \tag{8}$$

Since for any $t \in [-\pi, \pi]$

$$\begin{aligned}(S_n\varphi)(t) &= \frac{1}{n}\sum_{k=0}^{n-1} \cot\left(-\frac{\pi(2k+1)}{2n}\right)\varphi\left(t + \frac{\pi(2k+1)}{n}\right)\\ &= \frac{1}{n}\sum_{k\in\Delta_n} \cot\left(-\frac{\pi(2k+1)}{2n}\right)\varphi\left(t + \frac{\pi(2k+1)}{n}\right),\end{aligned}$$

$$(H_{2\pi/n}u)(t) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+\frac{\pi(2k+1)}{n})}{-k-1/2} = \frac{1}{\pi}\sum_{k\in\Delta_n} \frac{u(t+\frac{\pi(2k+1)}{n})}{-k-1/2} + \frac{1}{\pi}\sum_{k\in\tilde{\Delta}_n} \frac{u(t+\frac{\pi(2k+1)}{n})}{-k-1/2},$$

where

$$\begin{aligned}\Delta_n &= \left\{k \in \mathbb{Z} : \left[\frac{-n+1}{2}\right] \le k \le \left[\frac{n-1}{2}\right]\right\},\\ \tilde{\Delta}_n &= \left\{k \in \mathbb{Z} : |k| \le 2n,\ k > \left[\frac{n-1}{2}\right] \quad \text{or} \quad k < \left[\frac{-n+1}{2}\right]\right\},\end{aligned}$$

then for any $t \in [-\pi, \pi]$ we have

$$\begin{aligned}&(H_{2\pi/n}u)(t) - (S_n\varphi)(t) =\\ &= \frac{1}{n}\sum_{k\in\Delta_n}\left[\cot\frac{\pi(2k+1)}{2n} - \frac{2n}{\pi(2k+1)}\right]\varphi\left(t + \frac{\pi(2k+1)}{n}\right) + \frac{1}{\pi}\sum_{k\in\tilde{\Delta}_n}\frac{u\left(t+\frac{\pi(2k+1)}{n}\right)}{-k-1/2}.\end{aligned} \tag{9}$$

It follows from (9) and from inequality $|\cot x - 1/x| \le 2/\pi$ for $0 < |x| \le \pi/2$ that

$$\begin{aligned}&\|H_{2\pi/n}u - S_n\varphi\|_{L_p([-\pi,\pi])} \le\\ &\le \frac{1}{n}\sum_{k\in\Delta_n}\frac{2}{\pi}\|\varphi\|_{L_p(T)} + \frac{1}{\pi}\sum_{k\in\tilde{\Delta}_n}\frac{2}{n}\|\varphi\|_{L_p(T)} \le 4\|\varphi\|_{L_p(T)}.\end{aligned} \tag{10}$$

From (8) and (10) we have

$$\begin{aligned}\|S_n\varphi\|_{L_p(T)} &\le \|H_{2\pi/n}u - S_n\varphi\|_{L_p([-\pi,\pi])} + \|H_{2\pi/n}u\|_{L_p(\mathbb{R})}\\ &\le \left(4 + 2\|\tilde{h}\|_{l_p\to lp}\right)\cdot\|\varphi\|_{L_p(T)}.\end{aligned}$$

□

THEOREM 4. *The sequence of operators S_n strongly converges to the operator S in $L_p(T)$, $1 < p < \infty$, and for any $\varphi \in L_p(T)$ the following estimate holds:*

$$\|S\varphi - S_n\varphi\|_{L_p(T)} \leq \left(4 + \|S\|_{L_p(T)\to L_p(T)} + 2\|\tilde{h}\|_{l_p\to lp}\right) \cdot E^p_{n-1}(\varphi), \; n \in \mathbb{N}, \tag{11}$$

where $E^p_{n-1}(\varphi)$ – is the best approximation of the function φ in the metric $L_p(T)$ by trigonometric polynomials of order at most $n-1$, $n \in \mathbb{N}$.

Proof. Suppose that

$$q_{n-1}(t) = \tfrac{a_0}{2} + \sum_{m=1}^{n-1} (a_m \cos mt + b_m \sin mt)$$

is the best approximation of the function φ in the metric $L_p(T)$ by trigonometric polynomials of order at most $n-1$, $n \in \mathbb{N}$. Then it follows from the equality

$$(S_n q_{n-1})(t) = (S q_{n-1})(t)$$

that

$$(S\varphi - S_n\varphi)(t) = S(\varphi - q_{n-1})(t) - S_n(\varphi - q_{n-1})(t).$$

Then

$$\begin{aligned} \|S\varphi - S_n\varphi\|_{L_p(T)} &\leq \left(\|S\|_{L_p(T)\to L_p(T)} + \|S_n\|_{L_p(T)\to L_p(T)}\right) \cdot \|\varphi - q_{n-1}\|_{L_p(T)} \\ &\leq \left(4 + \|S\|_{L_p(T)\to L_p(T)} + 2\|\tilde{h}\|_{l_p\to lp}\right) \cdot E^p_{n-1}(\varphi). \end{aligned}$$

□

4. APPROXIMATION OF THE HILBERT TRANSFORM

Consider the regular integral operator

$$(\mathrm{K}\varphi)(t) = \tfrac{1}{2\pi} \int_{-\pi}^{\pi} K(t,\tau)\varphi(\tau)d\tau, \quad t \in T,$$

where $K(t,\tau)$ is a continuous function on $[-\pi,\pi]^2$, and the sequence of operators

$$(\mathrm{K}_n\varphi)(t) = \tfrac{1}{n} \sum_{k=0}^{n-1} K\left(t, t + \tfrac{\pi(2k+1)}{n}\right) \varphi\left(t + \tfrac{\pi(2k+1)}{n}\right), \quad t \in T, \quad n \in \mathbb{N},$$

where $K(t,\tau) = K(t, \tau - 2\pi)$ for $(t,\tau) \in [-\pi,\pi] \times (\pi, 3\pi)$.

LEMMA 5. *The sequence of operators $\{\mathrm{K}_n\}$ strongly converges to the operator* K *in $L_p(T)$.*

Proof. First assume that $K(t,\tau)$ is a 2π-periodic function by τ . Denote

$$\|K\|_\infty = \max_{t,\tau\in[-\pi,\pi]} |K(t,\tau)|, \quad E_n(K) = \inf \|K - \Phi_n\|_\infty,$$

where $\Phi_n(t,\tau)=\frac{\alpha_0(t)}{2}+\sum_{m=1}^{n}(\alpha_m(t)\cos m\tau+\beta_m(t)\sin m\tau)$, and infimum is taken over all trigonometric polynomials $\alpha_m(t)$, $m=\overline{0,n}$, $\beta_m(t)$, $m=\overline{1,n}$ of order at most n.

Denote $n_0=\left[\frac{n-1}{2}\right]$. Suppose that

$$q_{n_0}(t)=\tfrac{a_0}{2}+\sum_{m=1}^{n_0}(a_m\cos mt+b_m\sin mt)$$

and

$$\Phi_{n_0}^{(0)}(t,\tau)=\frac{\alpha_0^{(0)}(t)}{2}+\sum_{m=1}^{n_0}(\alpha_m^{(0)}(t)\cos m\tau+\beta_m^{(0)}(t)\sin m\tau)$$

are the best approximations of the functions φ and K by trigonometric polynomials of order at most n_0.

For any trigonometric polynomial $r_{n-1}(t)$ of order at most $n-1$, the equality

$$\tfrac{1}{2\pi}\int_{-\pi}^{\pi}r_{n-1}(\tau)d\tau=\tfrac{1}{n}\sum_{k=0}^{n-1}r_{n-1}\left(t+\tfrac{\pi(2k+1)}{n}\right)$$

holds. Therefore

$$\begin{aligned}&(\mathrm{K}\varphi)(t)-(\mathrm{K}_n\varphi)(t)=\\&=(\mathrm{K}-\mathrm{K}_n)(\varphi-q_{n_0})(t)+\tfrac{1}{2\pi}\int_{-\pi}^{\pi}\left[K(t,\tau)-\Phi_{n_0}^{(0)}(t,\tau)\right]q_{n_0}(\tau)d\tau\\&\quad+\tfrac{1}{n}\sum_{k=0}^{n-1}\left[K(t,t+\tau_k^{(n)})-\Phi_{n_0}^{(0)}(t,t+\tau_k^{(n)})\right]q_{n_o}(t+\tau_k^{(n)}),\end{aligned}$$

where $\tau_k^{(n)}=\frac{\pi(2k+1)}{n}$, $k\in\mathbb{Z}$. It follows from here and from inequalities

$$\|\mathrm{K}\|_{L_p(T)\to L_p(T)}\le\|K\|_\infty,\quad \|\mathrm{K}_n\|_{L_p(T)\to L_p(T)}\le\|K\|_\infty$$

that

$$\|\mathrm{K}\varphi-\mathrm{K}_n\varphi\|_{L_p(T)}\le 2\|K\|_\infty E_{n_0}^p(\varphi)+2E_{n_0}(K)\left[\|\varphi\|_{L_p(T)}+E_{n_0}^p(\varphi)\right].$$

This completes the proof of the lemma in this case. Now consider the general case.

Let $\varphi\in L_p(T)$ and $\varepsilon>0$. Denote

$$\begin{aligned}K^*(t,\tau)&=K(t,\tau)\ \text{ for }\ (t,\tau)\in[-\pi,\pi]\times[-\pi,\pi-\delta_\varepsilon],\\K^*(t,\tau)&=K(t,\pi-\delta_\varepsilon)+\tfrac{\tau-\pi+\delta_\varepsilon}{\delta_\varepsilon}[K(t,-\pi)-K(t,\pi-\delta_\varepsilon)]\\&\qquad\text{for }\ (t,\tau)\in[-\pi,\pi]\times[\pi-\delta_\varepsilon,\pi],\\K^*(t,\tau+2\pi)&=K^*(t,\tau)\ \text{ for any }(t,\tau)\in[-\pi,\pi]\times\mathbb{R},\end{aligned}$$

where $\delta_\varepsilon=\min\left\{2\pi\cdot\left(\frac{\varepsilon}{8\|K\|_\infty\|\varphi\|_{L_p(T)}}\right)^{\frac{p}{p-1}},\frac{\pi\varepsilon}{8\|K\|_\infty\|\varphi\|_{L_p(T)}},1\right\}$.

Since the function $K^*(t,\tau)$ is continuous and 2π-periodic by τ, the sequence of operators

$$(\mathrm{K}_n^*\varphi)(t)=\tfrac{1}{n}\sum_{k=0}^{n-1}K^*(t,t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}),\quad t\in T,\quad n\in\mathbb{N}$$

strongly converges to the operator

$$(\mathrm{K}^*\varphi)(t)=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}K^*(t,\tau)\varphi(\tau)d\tau$$

in $L_p(T)$. Therefore, the inequality

$$\|\mathrm{K}_n^*\varphi-\mathrm{K}^*\varphi\|_{L_p(T)}<\varepsilon/2$$

is satisfied for large values of n. Moreover, since

$$\begin{aligned}\|\mathrm{K}\varphi-\mathrm{K}^*\varphi\|_{L_p(T)}&\le\tfrac{1}{2\pi}\left[\int_{-\pi}^{\pi}\left(\int_{\pi-\delta_\varepsilon}^{\pi}|K(t,\tau)-K^*(t,\tau)||\varphi(\tau)|d\tau\right)^p dt\right]^{1/p}\\&\le\tfrac{\|K\|_\infty}{\pi}\left[\int_{-\pi}^{\pi}\left(\int_{\pi-\delta_\varepsilon}^{\pi}|\varphi(\tau)|d\tau\right)^p dt\right]^{1/p}\\&\le\tfrac{2\|K\|_\infty}{(2\pi)^{1-1/p}}(\delta_\varepsilon)^{1-1/p}\|\varphi\|_{L_p([\pi-\delta_\varepsilon,\pi])}\le\tfrac{\varepsilon}{4},\end{aligned}$$

and for $n\ge\frac{16\|K\|_\infty\|\varphi\|_{L_p(T)}}{\varepsilon}$

$$\|\mathrm{K}_n\varphi-\mathrm{K}_n^*\varphi\|_{L_p(T)}\le\tfrac{1}{n}\cdot\left(\tfrac{n}{2\pi}\cdot\delta_\varepsilon+1\right)\cdot2\|K\|_\infty\|\varphi\|_{L_p(T)}\le\tfrac{\varepsilon}{4},$$

then for sufficiently large values n we have

$$\begin{aligned}&\|\mathrm{K}_n\varphi-\mathrm{K}\varphi\|_{L_p(T)}\le\\&\le\|\mathrm{K}_n\varphi-\mathrm{K}_n^*\varphi\|_{L_p(T)}+\|\mathrm{K}_n^*\varphi-\mathrm{K}^*\varphi\|_{L_p(T)}+\|\mathrm{K}^*\varphi-\mathrm{K}\varphi\|_{L_p(T)}<\varepsilon.\end{aligned}$$

□

Corollary 6. *The sequence of operators*

$$(\widetilde{\mathrm{K}}_n\varphi)(t)=\tfrac{1}{n}\sum_{\{k\in\mathbb{Z}:t+\tau_k^{(n)}\in[-\pi,\pi]\}}K(t,t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}),\ t\in[-\pi,\pi],\ n\in\mathbb{N}$$

strongly converges to the operator K *in* $L_p([-\pi,\pi])$.

Corollary 7. *If the function* $K(t,\tau)$ *is continuous on* $[\pi m,\pi m+2\pi q]\times[-\pi,\pi]$, *then for any* $\varphi\in L_p(T)$ *the sequence of functions*

$$(\widetilde{\mathrm{K}}_n\varphi)(t)=\tfrac{1}{n}\sum_{\left\{k\in\mathbb{Z}:t+\tau_k^{(n)}\in[-\pi,\pi]\right\}}K(t,t+\tau_k^{(n)})\varphi(t+\tau_k^{(n)}),\ t\in[\pi m,\pi m+2\pi q],$$

converges to the function

$$(\mathrm{K}\varphi)(t)=\tfrac{1}{2\pi}\int_{-\pi}^{\pi}K(t,\tau)\varphi(\tau)d\tau,\quad t\in[\pi m,\pi m+2\pi q]$$

in $L_p([\pi m,\pi m+2\pi q])$, *where* $m\in\mathbb{Z}$, $q\in\mathbb{N}$.

COROLLARY 8. *If the function $K_0(t)$ is continuous on $[-\pi,\pi]$, then the sequence of operators*

$$(\mathrm{K}_n^0\varphi)(t)=\frac{1}{2n}\sum_{k=-n}^{n-1}K_0\left(\frac{\pi(2k+1)}{2n}\right)\varphi\left(t+\frac{\pi(2k+1)}{2n}\right),\ t\in T,\ n\in\mathbb{N}$$

strongly converges to the operator

$$(\mathrm{K}^0\varphi)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}K_0(\tau)\varphi(t+\tau)d\tau,\quad t\in T$$

in $L_p(T)$.

In the following theorem we prove that for any $\delta>0$ the sequence of operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_p(\mathbb{R})$, $1<p<\infty$.

THEOREM 9. *For any $\delta>0$ the sequence of the operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_p(\mathbb{R})$, that is for any $u\in L_p(\mathbb{R})$ the following inequality holds:*

$$\lim_{n\to\infty}\|H_{\delta/n}u-Hu\|_{L_p(\mathbb{R})}=0.$$

Proof. For simplicity of presentation we have divided the proof into three steps.

Step 1. Let us first prove that the operator

$$(H^*\varphi)(t)=\frac{1}{\pi}\int_{t-\pi}^{t+\pi}\frac{\varphi(\tau)}{t-\tau}d\tau$$

is a bounded operator in $L_p(T)$. Indeed, for any $\varphi\in L_p(T)$ we have

$$(H^*\varphi)(t)=\frac{1}{\pi}\int_{t-\pi}^{t+\pi}\frac{\varphi(\tau)}{t-\tau}d\tau=\frac{1}{\pi}\int_{t-\pi}^{t+\pi}\left[\frac{1}{t-\tau}-\frac{1}{2}\cot\frac{t-\tau}{2}\right]\varphi(\tau)d\tau+(S\varphi)(t)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\cot\frac{\tau}{2}-\frac{2}{\tau}\right]\varphi(t+\tau)d\tau+(S\varphi)(t).\tag{12}$$

Since the function

$$K_0(\tau)=\cot\frac{\tau}{2}-\frac{2}{\tau}\ \text{ for }\ \tau\neq 0,\quad K_0=0$$

is continuous on $[-\pi,\pi]$, then it follows from (12) and from Corollary 8 that the operator H^* is bounded in $L_p(T)$.

Consider the sequence of operators

$$(H_n^*\varphi)(t)=\frac{1}{\pi}\sum_{k=-n}^{n-1}\frac{1}{-k-1/2}\varphi\left(t+\frac{\pi(2k+1)}{2n}\right),\ t\in T,\ n\in\mathbb{N}.$$

Since for any $\varphi\in L_p(T)$

$$(H_n^*\varphi)(t)=\frac{1}{2n}\sum_{k=-n}^{n-1}\left[\cot\left(\frac{\pi(2k+1)}{4n}\right)-\frac{4n}{\pi(2k+1)}\right]\varphi\left(t+\frac{\pi(2k+1)}{2n}\right)+(S_{2n}\varphi)(t)=$$

$$= \frac{1}{2n} \sum_{k=-n}^{n-1} K_0\left(\frac{\pi(2k+1)}{2n}\right) \varphi\left(t + \frac{\pi(2k+1)}{2n}\right) + (S_{2n}\varphi)(t),$$

then it follows from Theorem 4 and from Corollary 8 that the sequence of operators H_n^* strongly converges to the operator H^* in $L_p(T)$.

Step 2. Let us first prove that the sequence of operators

$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi} \sum_{k\in\mathbb{Z}} \frac{1}{-k-1/2} u\left(t + \frac{\pi(k+1/2)}{4n}\right), \; t \in \mathbb{R}, \; n \in \mathbb{N}$$

strongly converges to the operator H in $L_p(\mathbb{R})$. At first assume that supp $u \subset [-\pi/4, \pi/4]$. Denote by φ 2π-periodic function, coinciding with the function u on $[-\pi/4, \pi/4]$ and equal to zero in $T \setminus [-\pi/4, \pi/4]$. Since for any $t \in [-\pi/2, \pi/2]$

$$(Hu)(t) = \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} \frac{u(\tau)}{t-\tau} d\tau = (H^*\varphi)(t), \tag{13}$$

$$\begin{aligned}(H_{\pi/n}u)(t) &= \frac{1}{\pi} \sum_{k=-n}^{n-1} \frac{1}{-k-1/2} u\left(t + \frac{\pi(k+1/2)}{n}\right) \\ &= \frac{1}{\pi} \sum_{k=-n}^{n-1} \frac{1}{-k-1/2} \varphi\left(t + \frac{\pi(k+1/2)}{n}\right) = (H_n^*\varphi)(t),\end{aligned} \tag{14}$$

and the sequence of operators H_n^* strongly converges to the operator H^* in $L_p(T)$, then it follows from (13) and (14) that for any $\varepsilon > 0$ for large values of n

$$\begin{aligned}\|H_{\pi/n}u - Hu\|_{L_p([-\pi/2,\pi/2])} &= \|H_n^*\varphi - H^*\varphi\|_{L_p([-\pi/2,\pi/2])} \\ &\le \|H_n^*\varphi - H^*\varphi\|_{L_p(T)} < \varepsilon.\end{aligned} \tag{15}$$

Due to the inequalities

$$|(Hu)(t)| \le \frac{1}{\pi} \int_{-\pi/4}^{\pi/4} \left|\frac{u(\tau)}{t-\tau}\right| d\tau \le \frac{\|u\|_{L_1([-\pi/4,\pi/4])}}{\pi(|t|-\pi/4)}, \; |t| > \pi/4,$$

$$\begin{aligned}|(H_{\pi/n}u)(t)| &\le \frac{1}{\pi} \sum_{k\in Z_{(n)}^{(t)}} \frac{1}{|k+1/2|} \left|u\left(t + \frac{\pi(k+1/2)}{n}\right)\right| \\ &\le \frac{1}{n(|t|-\pi/4)} \sum_{k\in Z_{(n)}^{(t)}} \left|u\left(t + \frac{\pi(k+1/2)}{n}\right)\right|, \quad |t| > \pi/4,\end{aligned}$$

where $Z_{(n)}^{(t)} = \{k \in \mathbb{Z} : t + \frac{\pi(k+1/2)}{n} \in [-\pi/4, \pi/4]\}$, we get that for any $M > 2\pi$

$$\|Hu\|_{L_p([M,\infty])} \le \frac{\|u\|_{L_1([-\pi/4,\pi/4])}}{\pi} \cdot \left(\int_M^\infty \frac{dt}{(|t|-\pi/4)^p}\right)^{1/p} = \frac{\|u\|_{L_1([-\pi/4,\pi/4])}}{\pi(p-1)^{1/p}(M-\pi/4)^{1-1/p}},$$

$$\|H_{\pi/n}u\|_{L_p([M,\infty])} \le \frac{1}{n}\left[\int_M^\infty \frac{1}{(|t|-\frac{\pi}{4})^p}\left(\sum_{k\in Z_{(n)}^{(t)}}\left|u(t+\frac{\pi(k+1/2)}{n})\right|\right)^p dt\right]^{1/p}$$

$$\le \frac{1}{n^{1/p}}\left[\int_M^\infty \frac{1}{(|t|-\frac{\pi}{4})^p}\sum_{k\in Z_{(n)}^{(t)}}\left|u\left(t+\frac{\pi(k+1/2)}{n}\right)\right|^p dt\right]^{1/p}$$

$$= \frac{1}{n^{1/p}}\left[\sum_{m=0}^\infty \int_{M+\frac{\pi m}{n}}^{M+\frac{\pi(m+1)}{n}} \frac{1}{(|t|-\frac{\pi}{4})^p}\sum_{k\in Z_{(n)}^{(t)}}\left|u\left(t+\frac{\pi(k+1/2)}{n}\right)\right|^p dt\right]^{1/p}$$

$$\le \frac{1}{n^{1/p}}\left[\sum_{m=0}^\infty \frac{1}{(M+\frac{\pi m}{n}-\frac{\pi}{4})^p}\int_{M+\frac{\pi m}{n}}^{M+\frac{\pi(m+1)}{n}} \sum_{k\in Z_{(n)}^{(t)}}\left|u\left(t+\frac{\pi(k+1/2)}{n}\right)\right|^p dt\right]^{1/p}$$

$$= \frac{1}{n^{1/p}}\left[\sum_{m=0}^\infty \frac{1}{(M+\frac{\pi m}{n}-\frac{\pi}{4})^p}\|u\|_{L_p([-\pi/4,\pi/4])}\right]^{1/p}$$

$$\le \frac{\|u\|_{L_p([-\pi/4,\pi/4])}}{n^{1/p}}\left[\frac{n/\pi}{(p-1)(M-\frac{\pi}{4}-\frac{\pi}{n})^{p-1}}\right]^{1/p}$$

$$= \frac{\|u\|_{L_p([-\pi/4,\pi/4])}}{\pi^{1/p}(p-1)^{1/p}(M-\frac{\pi}{4}-\frac{\pi}{n})^{1-1/p}}.$$

Similar inequalities holds for $\|Hu\|_{L_p([-\infty,-M])}$ and for $\|H_{\pi/n}u\|_{L_p([-\infty,-M])}$. Therefore, for any $\varepsilon>0$ there exist $m_0\ge 4$ such that

$$(16)\qquad \|Hu\|_{L_p(R\setminus[-\frac{\pi m_0}{2},\frac{\pi m_0}{2}])}<\varepsilon,\quad \|H_{\pi/n}u\|_{L_p(R\setminus[-\frac{\pi m_0}{2},\frac{\pi m_0}{2}])}<\varepsilon.$$

Since the function $\frac{1}{t-\tau}$ is continuous on a rectangle $[2\pi,2\pi m_0]\times[-\pi,\pi]$, then it follows from Corollary 7 that the sequence of functions

$$(\mathrm{W}_n\varphi)(t)=\frac{2}{n}\sum_{\{k\in\mathbb{Z}:t+\frac{\pi(2k+1)}{n}\in[-\pi,\pi]\}}\frac{\varphi(t+\pi(2k+1)/n)}{-\pi(2k+1)/n}=$$

$$=\frac{1}{\pi}\sum_{\{k\in\mathbb{Z}:t+\frac{\pi(2k+1)}{n}\in[-\pi,\pi]\}}\frac{\varphi(t+\pi(2k+1)/n)}{-k-1/2},\quad n\in\mathbb{N}$$

converges to the function

$$(\mathrm{W}\varphi)(t)=\int_{-\pi}^{\pi}\frac{\varphi(\tau)}{t-\tau}d\tau$$

in $L_p([2\pi,2\pi m_0])$. Denote by ψ the function, defined on $[-\pi,\pi]$ by the equality $\psi(\tau)=u(\tau/4)$. Then it follows from the equations

$$(Hu)(t)=\frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{u(\tau)}{t-\tau}d\tau=\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{u(\tau/4)}{4t-\tau}d\tau=(\mathrm{W}\psi)(4t),\ \ t\in[\pi/2,\pi m_0/2],$$

$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi}\sum_{k\in Z^{(t)}_{(4n)}} \frac{u(t+\pi(k+1/2)/4n)}{-k-1/2} = \frac{1}{\pi}\sum_{k\in Z^{(t)}_{(4n)}} \frac{\psi(4t+\pi(k+1/2)/n)}{-k-1/2}$$
$$= (\mathrm{W}_n\psi)(4t), \quad t \in [\pi/2, \pi m_0/2],$$

that the sequence of functions $H_{\pi/(4n)}u$ converges to the function Hu in $L_p([\pi/2, \pi m_0/2])$. Therefore, for large values of n

(17) $$\|H_{\pi/(4n)}u - Hu\|_{L_p([\pi/2,\pi m_0/2])} < \varepsilon.$$

Similarly, for large values on n

(18) $$\|H_{\pi/(4n)}u - Hu\|_{L_p([-\pi m_0/2,-\pi/2])} < \varepsilon.$$

It follows from (15)–(18) that in the case $\operatorname{supp} u \subset [-\pi/4, \pi/4]$

(19) $$\lim_{n\to\infty} \|H_{\pi/(4n)}u - Hu\|_{L_p(\mathbb{R})} = 0.$$

Now suppose that $\operatorname{supp} u \subset [-\pi m/4, \pi m/4]$ for some $m \in \mathbb{N}$. Denote by u_0 the function, defined on $[-\pi/4, \pi/4]$ by the equation $u_0(t) = u(mt)$. Then for any $t \in R$

$$(Hu)(t) = \frac{1}{\pi}\int_{-\pi m/4}^{\pi m/4} \frac{u(\tau)}{t-\tau}d\tau = \frac{1}{\pi}\int_{-\pi/4}^{\pi/4} \frac{u(m\tau)}{t-\tau} m d\tau = (Hu_0)(t/m),$$
$$(H_{\pi/(4n)}u)(t) = \frac{1}{\pi}\sum_{\{k\in\mathbb{Z}: t+\frac{\pi(k+1/2)}{4n}\in[-\frac{\pi m}{4},\frac{\pi m}{4}]\}} \frac{u(t+\pi(k+1/2)/4n)}{-k-1/2}$$
$$= \frac{1}{\pi}\sum_{k\in Z^{(t/m)}_{(4mn)}} \frac{u_0(t/m+\pi(k+1/2)/(4mn))}{-k-1/2} = (H_{\pi/(4mn)}u_0)(t/m).$$

Since equation (19) holds for u_0, we obtain that

$$\lim_{n\to\infty} \|H_{\pi/(4n)}u - Hu\|_{L_p(\mathbb{R})} = m^{1/p} \lim_{n\to\infty} \|H_{\pi/(4mn)}u_0 - Hu_0\|_{L_p(\mathbb{R})} = 0.$$

Now consider the general case. Let us prove that equation (19) holds for any $u \in L_p(\mathbb{R})$. For any $u \in L_p(\mathbb{R})$ and $\varepsilon > 0$ there exist $m \in \mathbb{N}$ such that

(20) $$\|u - u_m\|_{L_p(\mathbb{R})} < \varepsilon,$$

where $u_m(t) = u(t)\cdot\chi_{[-\pi m/4,\pi m/4](t)}$. Since equation (19) holds for u_m, and it follows from (3), (20) that

$$\|H_{\pi/(4n)}(u - u_m) - H(u - u_m)\|_{L_p(\mathbb{R})} \le$$
$$\le \left[\|H_{\pi/(4n)}\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})} + \|H\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})}\right]\cdot\|u - u_m\|_{L_p(\mathbb{R})}$$
$$\le \varepsilon\cdot\left[\|\tilde{h}\|_{l_p\to l_p} + \|H\|_{L_p(\mathbb{R})\to L_p(\mathbb{R})}\right],$$

then we get that the equation (19) also holds for the function u.

Step 3. Let us prove that for any $\delta > 0$ the secuence of the operators $\{H_{\delta/n}\}_{n\in\mathbb{N}}$ strongly converges to the operator H in $L_p(\mathbb{R})$. Let $u \in L_p(\mathbb{R})$. Denote $w(t) = u(4\delta t/\pi)$, $t \in \mathbb{R}$. Then for any $t \in \mathbb{R}$

$$(Hu)(t) = \frac{1}{\pi}\int_R \frac{u(\tau)}{t-\tau}d\tau = \frac{1}{\pi}\int_R \frac{w(\pi\tau/(4\delta))}{t-\tau}d\tau =$$

$$= \frac{1}{\pi}\int_R \frac{w(\tau)}{\pi t/(4\delta)-\tau}d\tau = (Hw)(\pi t/(4\delta)), \tag{21}$$

$$(H_{\delta/n}u)(t) = \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{u(t+(k+1/2)\delta/n)}{-k-1/2} =$$

$$= \frac{1}{\pi}\sum_{k\in\mathbb{Z}} \frac{w(\pi t/(4\delta)+\pi(k+1/2)/(4n))}{-k-1/2} = (H_{\pi/(4n)}w)(\pi t/(4\delta)). \tag{22}$$

Since $\lim_{n\to\infty} \|H_{\pi/(4n)}w - Hw\|_{L_p(\mathbb{R})} = 0$, then it follows from (21), (22) that

$$\lim_{n\to\infty} \|H_{\delta/n}u - Hu\|_{L_p(\mathbb{R})} = 0.$$

□

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