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POINTWISE COPROXIMINALITY IN $L^p(\mu, X)$

EYAD ABU-SIRHAN*

Abstract. Let X be a Banach space, G be a closed subspace of X, (Ω, Σ, μ) be a σ-finite measure space, $L(\mu, X)$ be the space of all strongly measurable functions from Ω to X, and $L^p(\mu, X)$ be the space of all Bochner p-integrable functions from Ω to X. The purpose of this paper is to discuss the relationship between the pointwise coproximinality of $L(\mu, G)$ in $L(\mu, X)$ and the pointwise coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$.

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1. INTRODUCTION

In this article, $(X, \|\cdot\|)$ will be referred to as a Banach space over the field R of real numbers, G as a closed subspace of X, and (Ω, Σ, μ) as a σ-finite nontrivial measure space, *i.e.* Ω is a countable union of measurable sets each with finite measure and there exists at least $A \in \Sigma$ with $\infty > \mu(A) > 0$. For $p \geq 1$, we suppose $L^p(\mu, X)$ is the space of all Bochner p-integrable functions from Ω to X, $L(\mu, X)$ is the linear space of all μ-equivalence classes of strongly measurable functions from Ω to X, see [10]. If a subset M of $L(\mu, X)$ is closed with regard to the pointwise limits of sequences, we say that it is closed. For $E \subseteq \Omega$ and a function $f : \Omega \to X$, χ_E is the characteristic function of the set E and $\chi_E \otimes f$ is the function $\chi_E \otimes f(s) = \chi_E(s)f(s)$. A function $f : \Omega \to X$ is said to be simple if

$$f = \sum_{i=1}^{n} x_i \chi_{E_i},$$

where $x_i \in X$ and $E_i = f^{-1}(x_i)$ is measurable for $i = 1, 2, \dots, n$. A function $f : \Omega \to X$ is said to be strongly measurable if there exists a sequence of simple functions $\{f_n\}$ with

$$\lim_{n\to\infty} \|f_n(s) - f(s)\| = 0, \qquad a.e.$$

Let M be a closed subspace of $L(\mu, X)$, we write $L^p(M)$ for the Banach space of all functions in M such that $\int_\Omega \|f(s)\|^p$ is finite. An element $g_0 \in G$ is said

*Department of Mathematics, Tafila Technical University, Jordan, e-mail: abu-sirhan2@ttu.edu.jo, https://orcid.org/0000-0003-1360-5554.

to be a best coapproximation of $x \in X$ if

$$\|g - g_0\| \leq \|x - g\| \qquad \forall g \in G.$$

The set of all elements of best coapproximation of x in G will be denoted by $R_G(x)$. If $R_G(x)$ is nonempty for any $x \in X$, G is said to be a coproximinal subspace of X. This new kind of approximation has been introduced by C. Franchetti and M. Furri (1972) [4] to characterize real Hilbert spaces among real reflexive Banach spaces.

P.L. Papini and I. Singer (1979) [8] then went into greater depth on the best coapproximation. It has lately been studied in $L^p(\mu, X)$; for example, see [1, 2, 6, 7, 5]. With finite measure spaces, though, [5, 6, 7] have dealt. This paper's goal is to demonstrate the connection between the pointwise coproximinality of $L^p(\mu, G)$ in $L^p(\mu, X)$ and the pointwise coproximinality of $L(\mu, G)$ in $L(\mu, X)$. Pointwise coproximinality is a counterpart of pointwise proximinality, it should be noted, see [9, 2].

2. POINTWISE COPROXIMINALITY

Definition 1. Let M be a subset of $L(\mu, X)$ and $f \in L(\mu, X)$. An element ϕ_0 in M is said to be a best pointwise coapproximation of f from M if for any $\phi \in M$,

$$\|\phi_0(s) - \phi(s)\| \leq \|f(s) - \phi(s)\| \qquad a.e.$$

M is said to be pointwise coproximinal in $L(\Omega, X)$ if each element of $L(\Omega, X)$ has a best pointwise coapproximation from M.

Lemma 1. *Let M be a subset of $L(\mu, X)$, $f \in M$, and $A \in \Sigma$. If M is pointwise coproximinal in $L(\mu, X)$, then $\chi_A \otimes f \in M$.*

Proof. Assume that M is pointwise coproximinal and there exist $A \in \Sigma$, $f \in M$, such that $\chi_A \otimes f \notin M$. By assumption there exists ϕ_A in M such that

$$\|\phi_A(s) - \phi(s)\| \leq \|\chi_A \otimes f(s) - \phi(s)\| \qquad a.e.$$

and for all $\phi \in M$. Taking $\phi = 0$ one obtains

$$\begin{aligned} \|\phi_A(s)\| &\leq \|\chi_A \otimes f(s)\| \\ &= \chi_A(s)\|f(s)\| \qquad a.e. \end{aligned} \tag{1}$$

For $\phi = f$ one obtains

$$\begin{aligned} \|\phi_A(s) - f(s)\| &\leq \|\chi_A \otimes f(s) - f(s)\| \\ &= \chi_{A^c}(s)\|f(s)\| \qquad a.e. \end{aligned} \tag{2}$$

By (1), $\phi_A(s) = 0 \quad a.e.\ s \in A^c$ and by (2) $\phi_A(s) = f(s) \quad a.e.\ s \in A$, that is, $\chi_A \otimes f = \phi_A \in M$. The proof is completed by the contradiction. □

Corollary 1. *Let M be a closed subspace of $L(\mu, X)$. If M is pointwise coproximinal in $L(\mu, X)$, then M is an $L(\mu, R)$-submodule of $L(\mu, X)$.*

Proof. The proof follows from Lemma 1 and the fact that the set of all simple functions in $L(\mu, X)$ is dense in $L(\mu, X)$.

Let $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ be a family of countable subset of Ω. if $A_n \in \Sigma$ for any $n \in \mathbb{N}$, $A_i \cap A_j = \phi$ for $i \neq j$, and $\mu(\Omega - \cup_{n\in\mathbb{N}} A_n) = 0$, then the set $\mathcal{A}$ is said to be measurable partition of Ω [3]. □

Lemma 2 ([2]). *Let G be a closed subspace of X, $1 \leq p \leq \infty$, $f \in L^p(\mu, X)$, and $h \in L(\mu, X)$. If h is pointwise coapproximation to f, then $h \in L^p(\mu, G)$ and it is a best coapproximation to f from $L^p(\mu, G)$.*

Lemma 3. *Let M be a closed subspace of $L(\mu, X)$ and $f \in M$. Then there exists a measurable partition $\{D_n, n \in \mathbb{N}\}$ such that*

$$f(s) = \sum_{n\in\mathbb{N}} \chi_{D_n} \otimes f(s)$$

and $\chi_{D_n} \otimes f \in L^p(M)$, $1 \leq p \leq \infty$, for all n in $\mathbb{N}$.

Proof. Since (Ω, Σ, μ) is σ-finite, we may assume that $\Omega = \bigcup_{n\in\mathbb{N}} A_n$, A_n is measurable, $A_n \subseteq A_{n+1}$, and $\mu(A_n) < \infty$ for any $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, let

$$B_n = \{s \in \Omega : \|f(s)\| \leq n\},$$

$C_n = A_n \cap B_n$, and $D_n = C_n - C_{n-1}$. Then $\{D_n, n \in \mathbb{N}\}$ is a measurable partition, $\chi_{D_n} \otimes f \in L^p(M)$ by Lemma 1, and $f = \sum_{n=1}^{\infty} \chi_{D_n} \otimes f$. □

Theorem 1. *Let G be a closed subspace of X. Then the following are equivalent*

(1) $L(\mu, G)$ is pointwise coproximinal in $L(\mu, X)$.

(2) $L^p(\mu, G)$ is pointwise coproximinal in $L^p(\mu, X)$.

Proof. (1) $\Rightarrow$ (2). Let$f \in L^p(\mu, X)$. Then $f \in L(\mu, X)$ and it has a best pointwise coapproximation f_0 from $L(\mu, G)$. By Lemma 2, f_0 is a best pointwise coapproximation to f from $L^p(\mu, G)$.

(2) $\Rightarrow$ (1). Let $f \in L(\mu, X)$. By Lemma 3, there exists $\{D_n : n \in \mathbb{N}\}$ measurable partition of Ω such that

$$f = \sum_{n\in\mathbb{N}} \chi_{D_n} \otimes f$$

and $\chi_{D_n} \otimes f \in L^p(\mu, X)$ for any n in $\mathbb{N}$ and $1 \leq p \leq \infty$. By assumption, there exists $\phi_n \in L^p(\mu, G)$ such that

$$\|\phi_n(s) - \phi(s)\| \leq \|\chi_{D_n} \otimes f(s) - \phi(s)\| \qquad a.e. \tag{3}$$

for all $\phi \in L(\mu, G)$. Taking $\phi = 0$ yields,

$$\|\phi_n(s)\| \leq \|\chi_{D_n} \otimes f(s)\| \qquad a.e.$$

and hence $\phi_n = \chi_{D_n} \otimes \phi_n$. If we let $\bar{\phi} = \sum_{n\in\mathbb{N}} \phi_n$, then $\bar{\phi} \in L(\mu, G)$. We claim that

$$\|\bar{\phi}(s) - \phi(s)\| \leq \|f(s) - \phi(s)\| \qquad a.e.$$

for all $\phi \in L(\mu, G)$. If the claim is incorrect, then $\phi_0 \in L(\mu, G)$ exists such that

$$\mu\{s : \|\bar{\phi}(s) - \phi_0(s)\| > \|f(s) - \phi_0(s)\|\} > 0$$

and let $A_0 = \{s : \|\bar{\phi}(s) - \phi_0(s)\| > \|f(s) - \phi_0(s)\|\}$. By Lemma 3, there exists a measurable partition $\{C_n : n \in \mathbb{N}\}$ of Ω such that

$$\phi_0 = \sum_{n \in \mathbb{N}} \chi_{C_n} \otimes \phi_0$$

and $\chi_{C_n} \otimes \phi_0 \in L^p(\mu, G)$ for any n in $\mathbb{N}$ and $1 \leq p \leq \infty$. Then $\{C_n \cap D_m : n, m \in \mathbb{N}\}$ is a measurable partition of Ω. Hence there exist $n_0, m_0 \in \mathbb{N}$ such that $\mu(A_0 \cap C_{m_0} \cap D_{n_0}) > 0$. If $s \in A_0 \cap C_{m_0} \cap D_{n_0}$, then

$$\|\bar{\phi}(s) - \phi_0(s)\| > \|f(s) - \phi_0(s)\|,$$
$$\|\chi_{D_{n_0}} \otimes \phi_n(s) - \chi_{C_{m_0}} \otimes \phi_0(s)\| > \|\chi_{D_{n_0}} \otimes f(s) - \chi_{C_{m_0}} \otimes \phi_0(s)\|.$$

Let

$$B_0 = \{s : \|\chi_{D_{n_0}} \otimes \phi_n(s) - \chi_{C_{m_0}} \otimes \phi_0(s)\| > \|\chi_{D_{n_0}} \otimes f(s) - \chi_{C_{m_0}} \otimes \phi_0(s)\|\}.$$

Then $A_0 \cap C_{m_0} \cap D_{n_0} \subseteq B_0$ and $\mu(B_0) > 0$, which contradicts (1). □

Theorem 2 ([2]). *If $L^p(\mu, G)$ is pointwise coproximinal in $L^p(\mu, X)$, then G is coproximinal in X.*

Corollary 2. *If $L(\mu, G)$ is coproximinal in $L(\mu, X)$, then G is coproximinal in X.*

Proof. It is obvious that pointwise coproximinality of $L(\mu, G)$ in $L(\mu, x)$ follows from coproximinality of $L(\mu, G)$ in $L(\mu, x)$. Then $L^p(\mu, G)$ is pointwise coproximinal in $L^p(\mu, X)$ by Theorem 1. Therefore $L^p(\mu, G)$ is coproximinal in $L^p(\mu, X)$ and the result follows from Theorem 2. □

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