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AN EXTENSION OF THE CHENEY-SHARMA OPERATOR OF THE FIRST KIND

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Abstract. We extend the Cheney-Sharma operators of the first kind using Stancu type technique and we study some approximation properties of the new operator. We calculate the moments, we study local approximation with respect to a K-functional and the preservation of the Lipschitz constant and order.

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1. INTRODUCTION

In 1964, Cheney and Sharma introduced a generalization of the Bernstein polynomials,

$$B_n(f;x) := \sum_{k=0}^{n} \binom{n}{k} f\left(\tfrac{n}{k}\right) x^k (1-x)^{n-k}, \tag{1}$$

based on Jensen's generalization of the binomial theorem. In this respect, considering $\beta > 0$, $u := x$, $v := 1 - x$, when $x \in [0,1]$ and $m := n \in \mathbb{N}$, the first identity of the Abel-Jensen formulas,

$$(u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1}[v+(m-k)\beta]^{m-k}, \tag{2}$$

$$(u+v+m\beta)^m = \sum_{k=0}^{m} \binom{m}{k} (u+k\beta)^{k} v[v+(m-k)\beta]^{m-k-1}, \tag{3}$$

$$(u+v)(u+v+m\beta)^{m-1} = \sum_{k=0}^{m} \binom{m}{k} u(u+k\beta)^{k-1} v[v+(m-k)\beta]^{m-k-1}, \tag{4}$$

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generates the Cheney-Sharma operator of the first kind, defined, for $f : [0,1] \to \mathbb{R}$, by (see, *e.g.*, [1], [3]):

$$P_n^{\beta}(f;x) = \sum_{k=0}^{n} p_{n,k}^{\beta}(x) f\left(\tfrac{k}{n}\right), \tag{5}$$

where

$$p_{n,k}^{\beta}(x) = \tbinom{n}{k} \tfrac{x(x+k\beta)^{k-1}[1-x+(n-k)\beta]^{n-k}}{(1+n\beta)^n}. \tag{6}$$

The following properties can be found in [3]. With the usual notations $e_k(t) := t^k$, $t \in [0,1]$, $k = 0,1,2,\ldots$, by direct calculation in (2), one easily obtains

$$P_n^{\beta}(e_0;x) = 1.$$

Furthermore, Cheney and Sharma highlighted that

$$P_n^{\beta}(e_1;x) = A_n x, \tag{7}$$

where

$$A_n = (1+n\beta)^{-1} \int_0^{\infty} e^{-t} \left(1 + \tfrac{t\beta}{1+n\beta}\right)^{n-1} dx. \tag{8}$$

Considering $\beta = o\left(\frac{1}{n}\right)$, $A_n \le 1$ and (A_n) tends to 1, there is obtained

$$P_n^{\beta}(e_1;x) \to x \text{ uniformly on } [0,1].$$

Applying a reduction formula, Cheney and Sharma also proved that

$$P_n^{\beta}(e_2;x) \to x^2 \text{ uniformly on } [0,1].$$

Since P_n^{β} is nonnegative for $0 \le \beta = o\left(\frac{1}{n}\right)$, applying the Korovkin theorem, it is obtained that for all $f \in C[0,1]$,

$$P_n^{\beta}(f;x) \to f \text{ uniformly on } [0,1].$$

For $\beta = 0$, one easily obtains the Bernstein operator,

$$P_n^0 = B_n.$$

In 1982, Stancu [6] introduced a new Bernstein type operator,

$$L_{n,r} = \sum_{k=0}^{n-r} b_{n-r,k}(x) \left[(1-x) f\left(\tfrac{k}{n}\right) + x f\left(\tfrac{k+r}{n}\right)\right],$$

where $b_{n,k}$ denote the basis Bernstein polynomials of degree n,

$$b_{n,k} = \tbinom{n}{k} x^k (1-x)^{n-k},\ k = 0,1,\ldots,n,$$

for $f \in C[0,1]$, $n, r \in \mathbb{N}$ such that $n > 2r$.

In the present paper we introduce *the Stancu type extension of the Cheney and Sharma operator of the first kind*, based on an ideea from [4], using the

Stancu type operator $L_{n,r}$, and the Cheney and Sharma operator of the first kind, P_n^{β}, that is given by

$$L_{P_{n,r}^{\beta}}(f;x) := \sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)f\left(\tfrac{k}{n}\right) + xf\left(\tfrac{k+r}{n}\right)\right], \tag{9}$$

where $p_{n-r,k}$ is given by (6), considering $n-r$ in places of n, $f \in C[0,1]$ and $n, r \in \mathbb{N}$ such that $n > 2r$.

In order to obtain approximation results, we consider, as in [3], $\beta \geq 0$ such that $\beta = o\left(\frac{1}{n}\right)$. By direct calculation, it is obtained that $L_{P_{n,r}^{0}}$ reduces to the Stancu operator $L_{n,r}$, while $L_{P_{n,0}^{\beta}}$ represents the Cheney-Sharma operator of the first kind, P_n^{β}.

As in [4], we are going to calculate the moments of the new operators, using a reduction formula from [3], and we study local approximation properties with respect to an appropriate K-functional. Moreover, we emphasize the preservation of the Lipschitz constant and the order when applying $L_{P_{n,r}^{\beta}}$ operator to a Lipschitz continuous function, in a similar manner to the one presented in [2] and [4].

2. PROPERTIES OF CHENEY-SHARMA OPERATOR OF THE FIRST KIND

The purpose of this section is to outline some intermediary results regarding the Cheney and Sharma operators of the first kind that will be used in the sequel in order to prove some properties of the new constructed operator.

LEMMA 1 ([3]). *Let $x, y \in [0,1]$, $n \in \mathbb{N}$, $k = 0,1,\ldots,n$. Then the function*

$$S(k,n,x,y) := \sum_{\mu=0}^{n} \binom{n}{\mu}(x+\mu\beta)^{\mu+k-1}(y+(n-\mu)\beta)^{n-\mu}$$

satisfies the recurrence relation

$$S(k,n,x,y) = S(k-1,n,x,y) + n\beta S(k,n-1,x+\beta,y). \tag{10}$$

COROLLARY 2 ([3]). *By applying recursively the formula (10), one obtains*

$$S(1,n,x,y) = \int_0^{\infty} e^{-t}(x+y+n\beta+t\beta)^n dt, \tag{11}$$

and

$$S(2,n,x,y) = \int_0^{\infty} e^{-t}dt \int_0^{\infty} e^{-s}ds\Big[x(x+y+n\beta+t\beta+s\beta)^n + n\beta^2 s(x+y+n\beta+t\beta+s\beta)^{n-1}\Big]. \tag{12}$$

The next result refers to the moments of the Cheney-Sharma operator of first kind.

Lemma 3 (see [3] for a) and b)). *For every* $x \in [0,1]$, $n \in \mathbb{N}$ *one obtains*

a) $P_n^\beta(e_0;x) = 1$,

b) $P_n^\beta(e_1;x) = A_n x$, *with* A_n *given by* (8),

c) $P_n^\beta(e_2;x) = \frac{n-1}{n}\left[x(x+2\beta)\tilde{A}_n + x(n-2)\beta^2\tilde{B}_n\right] + \frac{1}{n}A_n x$,

with $\tilde{A}_n$ *and* $\tilde{B}_n$ *given by*

$$\tilde{A}_n = \frac{1}{(1+n\beta)^2}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\left(1+\frac{t\beta+s\beta}{1+n\beta}\right)^{n-2} \tag{13}$$

$$\tilde{B}_n = \frac{1}{(1+n\beta)^3}\int_0^\infty e^{-t}dt\int_0^\infty se^{-s}ds\left(1+\frac{t\beta+s\beta}{1+n\beta}\right)^{n-3}. \tag{14}$$

Moreover, considering $0 \le \beta = o\left(\frac{1}{n}\right)$, *one determines*

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty}\tilde{A}_n = \lim_{n\to\infty}\tilde{B}_n = 1.$$

Proof. a) This result is easily obtained by direct calculation, starting from (2).

b) The second result is presented in [3].

c) For obtaining $P_n^\beta(e_2;x)$, we consider another method than the one used in [3], where the authors emphasized some bounds for the terms of $P_n^\beta(e_2;x)$ which are helpful in order to prove that $P_n^\beta(e_2;x)$ converges uniformly to x^2, taking into consideration that $0 \le \beta = o\left(\frac{1}{n}\right)$. But for calculating the moments of the new introduced operator, we need to highlight some coefficients of the monomials e_1 and e_2 that appear in $P_n^\beta(e_2;x)$.

We have [3]

$$\frac{k^2}{n^2} = \frac{n-1}{n}\cdot\frac{k}{n}\cdot\frac{k-1}{n-1} + \frac{k}{n^2},$$

that we replace in $P_n^\beta(e_2;x)$ and applying (12), we obtain

$$\begin{aligned} P_n^\beta(e_2;x) &= \frac{n-1}{n}(1+n\beta)^{-n}xS(2,n-2,x+2\beta,1-x) + \frac{1}{n}P_n^\beta(e_1;x) \\ &= \frac{n-1}{n}\left[x(x+2\beta)\tilde{A}_n + x(n-2)\beta^2\tilde{B}_n\right] + \frac{1}{n}A_n x, \end{aligned}$$

where $\tilde{A}_n$ and $\tilde{B}_n$ are given in (13) and (14), obtained by rewriting the integrals in (12). The first limit was evaluated in [3] using the double inequality,

$$e^{nu}(1-nu^2) \le (1+u)^n \le e^{nu}, \tag{15}$$

which will also be helpful for calculating the limits of $\tilde{A}_n$ and $\tilde{B}_n$.

Indeed, by applying the right inequality from (15), we obtain

$$\tilde{A}_n \le \frac{1}{(1+n\beta)^2}\int_0^\infty e^{-t}dt\int_0^\infty e^{-s}ds\; e^{\left(\frac{t\beta+s\beta}{1+n\beta}\right)(n-2)}$$

$$= \frac{1}{(1+n\beta)^2} \left(\int_0^\infty e^{-t\left(\frac{1+2\beta}{1+n\beta}\right)} dt \right)^2 = \frac{1}{(1+2\beta)^2}.$$

Similarly, there are obtained the following inequalities:

$$\frac{1}{(1+2\beta)^2} - \frac{2n\beta^2}{(1+2\beta)^4} \le \tilde{A}_n \le \frac{1}{(1+2\beta)^2},$$
$$\frac{1}{(1+3\beta)^3} - \frac{5n\beta^2}{3(1+3\beta)^5} \le \tilde{B}_n \le \frac{1}{(1+3\beta)^3}.$$

Considering $0 \le \beta = o\left(\frac{1}{n}\right)$, it is easily seen that $\lim_{n\to\infty} \tilde{A}_n = \lim_{n\to\infty} \tilde{B}_n = 1$, hence $P_n^\beta(e_2;x)$ converges uniformly to x^2. □

3. PROPERTIES OF THE STANCU TYPE EXTENSION OF THE CHENEY-SHARMA OPERATOR OF THE FIRST KIND

In this section we study some approximation properties for the new operator, $L_{P_{n,r}^\beta}$, introduced by us in (9).

First, we highlight the expressions of the moments of $L_{P_{n,r}^\beta}$ in terms of the moments of the Cheney-Sharma operators of the first kind.

Lemma 4. *For every $x \in [0,1]$, $n, r \in \mathbb{N}$ such that $n > 2r$, we obtain*

$$\begin{aligned} L_{P_{n,r}^\beta}(e_0;x) &= 1,\\ L_{P_{n,r}^\beta}(e_1;x) &= \tfrac{n-r}{n} A_{n-r} x + \tfrac{r}{n} x,\\ L_{P_{n,r}^\beta}(e_2;x) &= \tfrac{(n-1)(n-r-1)}{n^2}\left[x(x+2\beta)\tilde{A}_{n-r} + x(n-r-2)\beta^2 \tilde{B}_{n-r}\right]\\ &\quad + \tfrac{n-r}{n^2}(1+2xr)A_{n-r}x + \tfrac{r^2}{n^2}x. \end{aligned}$$

Proof. Taking into account (5), (6), (9), Lemma 3, in a similar manner to [4], we obtain

$$\begin{aligned} &L_{P_{n,r}^\beta}(e_0;x) = P_{n-r}^\beta(e_0;x) = 1,\\ &L_{P_{n,r}^\beta}(e_1;x) = \tfrac{n-r}{n} P_{n-r}^\beta(e_1;x) + \tfrac{r}{n} x P_{n-r}^\beta(e_0;x) = \tfrac{n-r}{n} A_{n-r} x + \tfrac{r}{n} x,\\ &L_{P_{n,r}^\beta}(e_2;x) = \tfrac{(n-r)^2}{n^2} P_{n-r}^\beta(e_2;x) + 2xr\tfrac{n-r}{n^2} P_{n-r}^\beta(e_1;x) + \tfrac{r^2}{n^2} x P_{n-r}^\beta(e_0;x)\\ &= \tfrac{(n-r)^2}{n^2}\left\{\tfrac{n-r-1}{n-r}[x(x+2\beta)\tilde{A}_{n-r} + x(n-r-2)\beta^2\tilde{B}_{n-r}] + \tfrac{1}{n-r}A_{n-r}x\right\}\\ &\quad + 2xr\tfrac{n-r}{n^2}A_{n-r}x + \tfrac{r^2}{n^2}x\\ &= \tfrac{(n-1)(n-r-1)}{n^2}\left[x(x+2\beta)\tilde{A}_{n-r} + x(n-r-2)\beta^2\tilde{B}_{n-r}\right]\\ &\quad + \tfrac{n-r}{n^2}(1+2xr)A_{n-r}x + \tfrac{r^2}{n^2}x. \end{aligned}$$

□

Lemma 5. *For every $f \in C[0,1]$, we have*

$$\|L_{P_{n,r}^\beta}\| \le \|f\|.$$

Proof. Considering the expression of $L_{P_{n,r}^{\beta}}(f;x)$ and Lemma 3, we get

$$\begin{aligned}|L_{P_{n,r}^{\beta}}(f;x)| &= \left|\sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)f\left(\tfrac{k}{n}\right)+xf\left(\tfrac{k+r}{n}\right)\right]\right| \\ &\le \sum_{k=0}^{n-r} p_{n-r,k}(x)\left|(1-x)f\left(\tfrac{k}{n}\right)+xf\left(\tfrac{k+r}{n}\right)\right| \\ &\le \sum_{k=0}^{n-r} p_{n-r,k}(x)\left[(1-x)\left|f\left(\tfrac{k}{n}\right)\right|+x\left|f\left(\tfrac{k+r}{n}\right)\right|\right] \\ &\le \|f\|\sum_{k=0}^{n-r} p_{n-r,k}(x)(1-x+x) \\ &= \|f\|P_{n-r}^{\beta}(e_0;x) \\ &= \|f\|.\end{aligned}$$

□

The next approximation result is based on the equivalence between the first order modulus of smoothness and the following K-functional,

$$K_1(f,\delta)=\inf_{g\in W^1}\left\{\|f-g\|+\delta\|g'\|\right\},$$

where $\delta>0$ and $W^1 := \{g\in C[0,1] : g'\in C[0,1]\}$.

Considering the first order modulus of smoothness associated to a function $f\in C[0,1]$, given by $\omega(f,\delta)=\sup_{0\le h\le\delta;x,x+h\in[0,1]}|f(x+h)-f(x)|$, in [7], it is asserted that there exists a positive constant $C>0$ such that

$$C^{-1}\cdot\omega(f,\delta)\le K_1(f,\delta)\le C\cdot\omega(f,\delta). \tag{16}$$

THEOREM 6. *Let $f\in C[0,1]$, $x\in[0,1]$ $n,r\in\mathbb{N}$ such that $n\ge 2r$, and $0\le\beta=o\left(\frac{1}{n}\right)$. Denoting*

$$\delta_{n,r}(x) := \left|L_{P_{n,r}^{\beta}}((e_1-x);x)\right| = \tfrac{n-r}{n}x(1-A_{n-r})$$

and

$$\tilde{\delta}_{n,r}(x)=\tfrac{\delta_{n,r}(x)}{2},$$

we obtain

$$|L_{P_{n,r}^{\beta}}(f;x)-f(x)|\le C\cdot\omega(f,\tilde{\delta}_{n,r}(x)).$$

Proof. For $g\in W^1$ and $x,t\in[0,1]$, by the Lagrange theorem, it results that there exists $c_{x,t}$ between x and t such that

$$g(t)-g(x)=g'(c_{x,t})(t-x).$$

Applying the operator $LP^{\beta}_{n,r}$ and taking into account the linearity of the operator, we get

$$\begin{aligned}|LP^{\beta}_{n,r}(g;x)-g(x)| &= |g'(c_{x,t})|\cdot|LP^{\beta}_{n,r}(e_1-x;x)|\\ &\leq \|g'\|\cdot\delta_{n,r}(x).\end{aligned}$$

Hence, by Lemma 5, we have

$$\begin{aligned}|LP^{\beta}_{n,r}(f;x)-f(x)| &= |LP^{\beta}_{n,r}(f-g;x)-(f-g)(x)+LP^{\beta}_{n,r}(g;x)-g(x)|\\ &\leq |LP^{\beta}_{n,r}(f-g;x)-(f-g)(x)|+|LP^{\beta}_{n,r}(g;x)-g(x)|\\ &\leq 2\|f-g\|+\|g'\|\cdot\delta_{n,r}(x)\\ &= 2\left(\|f-g\|+\tilde{\delta}_{n,r}(x)\cdot\|g'\|\right).\end{aligned}$$

Applying inequality (16), we obtain

$$|LP^{\beta}_{n,r}(f;x)-f(x)|\leq 2\cdot K_1(f,\tilde{\delta}_{n,r}(x))\leq C\cdot\omega(f,\tilde{\delta}_{n,r}(x)).$$

□

Next, we study the preservation of the Lipschitz constant and the order of a Lipschitz continuous function by applying the operator $LP^{\beta}_{n,r}$.

Theorem 7. *Let $f\in \mathrm{Lip}_M(\alpha,[0,1])$. Then*

$$LP^{\beta}_{n,r}f\in \mathrm{Lip}_M(\alpha,[0,1]),$$

for $n\in\mathbb{N}$, where $\mathrm{Lip}_M(\alpha,[0,1])$ denotes the class of Lipschitz continuous functions on $[0,1]$ of order $\alpha\in(0,1]$ and constant M, defined by

$$\mathrm{Lip}_M(\alpha,[0,1])=\Big\{f\in C[0,1]:|f(x)-f(y)|\leq M|x-y|^{\alpha}\Big\}.$$

Proof. The proof follows a similar manner to the one applied in [2] and [4]. Considering $x,y\in[0,1]$, $x\leq y$, we have

$$\begin{aligned}LP^{\beta}_{n,r}(f;y) &= \sum_{k=0}^{n-r}p_{n-r,k}(y)\left[(1-y)f\left(\tfrac{k}{n}\right)+yf\left(\tfrac{k+r}{n}\right)\right]\\ &= \tfrac{1}{(1+(n-r)\beta)^{n-r}}\sum_{k=0}^{n-r}\tbinom{n-r}{k}y(y+k\beta)^{k-1}\\ &\quad\cdot[1-y+(n-r-k)\beta]^{n-r-k}\left[(1-y)f\left(\tfrac{k}{n}\right)+yf\left(\tfrac{k+r}{n}\right)\right].\end{aligned}\tag{17}$$

Using the third Abel-Jensen identity (4), with $u:=x$, $v:=y-x$, $m:=k$, we obtain

$$y(y+k\beta)^{k-1}=\sum_{j=0}^{k}\tbinom{k}{j}x(x+j\beta)^{j-1}(y-x)[y-x+(k-j)\beta]^{k-j-1}.$$

Replacing this in (17), we get

$$L_{P^{\beta}_{n,r}}(f;y) = \frac{1}{(1+(n-r)\beta)^{n-r}} \sum_{j=0}^{n-r} \binom{n-r}{j} \sum_{k=0}^{j} \binom{j}{k} x(x+k\beta)^{k-1}(y-x)$$
$$\cdot [y-x+(j-k)\beta]^{j-k-1} [1-y+(n-r-j)\beta]^{n-r-j}$$
$$\cdot \left[(1-y)f\left(\frac{j}{n}\right) + yf\left(\frac{j+r}{n}\right)\right].$$

Denoting $l := j-k$, we have

$$L_{P^{\beta}_{n,r}}(f;y) = \frac{1}{(1+(n-r)\beta)^{n-r}} \sum_{k=0}^{n-r} \sum_{l=0}^{n-r-k} \binom{n-r}{k}\binom{n-r-k}{l} x(x+k\beta)^{k-1}(y-x)$$
$$\cdot (y-x+l\beta)^{l-1} [1-y+(n-r-k-l)\beta]^{n-r-k-l}$$
$$\cdot \left[(1-y)f\left(\frac{k+l}{n}\right) + yf\left(\frac{k+l+r}{n}\right)\right].$$

Now considering

(18)

$$L_{P^{\beta}_{n,r}}(f;x) = \frac{1}{(1+(n-r)\beta)^{n-r}} \sum_{k=0}^{n-r} \binom{n-r}{k} x(x+k\beta)^{k-1}$$
$$\cdot [1-x+(n-r-k)\beta]^{n-r-k} \left[(1-x)f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right],$$

by taking $u := y-x$, $v := 1-y$ and $m := n-r-k$ in the first Abel-Jensen identity (2), *i.e.*,

$$[1-x+(n-r-k)\beta]^{n-r-k} = \sum_{j=0}^{n-r-k} \binom{n-r-k}{j} (y-x)(y-x+j\beta)^{j-1}$$
$$\cdot [1-y+(n-r-k-j)\beta]^{n-r-k-j}$$

and replacing in (18), it results

$$L_{P^{\beta}_{n,r}}(f;x) = \frac{1}{(1+(n-r)\beta)^{n-r}} \sum_{k=0}^{n-r} \sum_{l=0}^{n-r-k} \binom{n-r}{k}\binom{n-r-k}{l} x$$
$$\cdot (x+k\beta)^{k-1}(y-x)(y-x+l\beta)^{l-1}[1-y+(n-r-k-l)\beta]^{n-r-k-l}$$
$$\cdot \left[(1-x)f\left(\frac{k}{n}\right) + xf\left(\frac{k+r}{n}\right)\right].$$

We obtain

$$L_{P^{\beta}_{n,r}}(f;y) - L_{P^{\beta}_{n,r}}(f;x) = \frac{1}{(1+(n-r)\beta)^{n-r}} \sum_{k=0}^{n-r} \sum_{l=0}^{n-r-k} \binom{n-r}{k}\binom{n-r-k}{l}$$
$$\cdot x(x+k\beta)^{k-1}(y-x)(y-x+l\beta)^{l-1} [1-y+(n-r-k-l)\beta]^{n-r-k-l}$$
$$\cdot \left[(1-y)\left(f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right)\right) + x\left(f\left(\frac{k+l+r}{n}\right) - f\left(\frac{k+r}{n}\right)\right)\right.$$

$$+(y-x)\left(f\left(\tfrac{k+l+r}{n}\right)-f\left(\tfrac{k}{n}\right)\right)\Big].$$

Using the hypothesis that $f\in \mathrm{Lip}_M(\alpha,[0,1])$, it results

$$\left|L_{P^{\beta}_{n,r}}(f;y)\ -L_{P^{\beta}_{n,r}}(f;x)\right| \le \tfrac{M}{(1+(n-r)\beta)^{n-r}}\sum_{k=0}^{n-r}\sum_{l=0}^{n-r-k}\tbinom{n-r}{k}\tbinom{n-r-k}{l}$$
$$\cdot\, x(x+k\beta)^{k-1}(y-x)(y-x+l\beta)^{l-1}\left[1-y+(n-r-k-l)\beta\right]^{n-r-k-l}$$
$$\cdot\left[(1-(y-x))\left(\tfrac{l}{n}\right)^{\alpha}+(y-x)\left(\tfrac{l+r}{n}\right)^{\alpha}\right],$$

and furthermore,

$$\left|L_{P^{\beta}_{n,r}}(f;y)\ -L_{P^{\beta}_{n,r}}(f;x)\right| \le \tfrac{M}{(1+(n-r)\beta)^{n-r}}\sum_{l=0}^{n-r}\tbinom{n-r}{l}(y-x)(y-x+l\beta)^{l-1}$$
$$\cdot\sum_{k=0}^{n-r-l}\tbinom{n-r-l}{k}x(x+k\beta)^{k-1}\left[1-y+(n-r-k-l)\beta\right]^{n-r-k-l}$$
$$\cdot\left[(1-(y-x))\left(\tfrac{l}{n}\right)^{\alpha}+(y-x)\left(\tfrac{l+r}{n}\right)^{\alpha}\right].$$

Using (2) with $u:=x$, $v:=1-y$ and $m:=n-r-l$, one obtains

$$[1-(y-x)+(n-r-l)\beta]^{n-r-l}=\sum_{k=0}^{n-r-l}\tbinom{n-r-l}{k}x(x+k\beta)^{k-1}$$
$$\cdot\,[1-y+(n-r-l-k)\beta]^{n-r-l-k}.$$

So we get,

$$\left|L_{P^{\beta}_{n,r}}(f;y)\ -\ L_{P^{\beta}_{n,r}}(f;x)\right| \le \tfrac{M}{(1+(n-r)\beta)^{n-r}}\sum_{l=0}^{n-r}\tbinom{n-r}{l}(y-x)(y-x+l\beta)^{l-1}$$
$$\cdot\,[1-(y-x)+(n-r-l)\beta]^{n-r-l}\left[(1-(y-x))\left(\tfrac{l}{n}\right)^{\alpha}+(y-x)\left(\tfrac{l+r}{n}\right)^{\alpha}\right].$$

Recall that $g(t)=t^{\alpha}$ is concave for $\alpha\in(0,1]$, meaning that

$$\alpha_1\cdot x_1^{\alpha}+\alpha_2\cdot x_2^{\alpha}\le(\alpha_1\cdot x_1+\alpha_2\cdot x_2)^{\alpha},\ \text{for } \alpha_1+\alpha_2=1.$$

In our case, $x_1:=\frac{l}{n}$, $x_2:=\frac{l+r}{n}$, $\alpha_1:=1-(y-x)$ and $\alpha_2:=y-x$, so it results

$$\left|L_{P^{\beta}_{n,r}}(f;y)\ -L_{P^{\beta}_{n,r}}(f;x)\right| \le \tfrac{M}{(1+(n-r)\beta)^{n-r}}\sum_{l=0}^{n-r}\tbinom{n-r}{l}(y-x)(y-x+l\beta)^{l-1}$$
$$\cdot\,[1-(y-x)+(n-r-l)\beta]^{n-r-l}\left[(1-(y-x))\,\tfrac{l}{n}+(y-x)\tfrac{l+r}{n}\right]^{\alpha}$$
$$=M\sum_{l=0}^{n-r}p_{n-r,l}(y-x)\left[(1-(y-x))\,\tfrac{l}{n}+(y-x)\tfrac{l+r}{n}\right]^{\alpha}.$$

Case $\alpha = 1$**.** The last inequality becomes

$$\begin{aligned}\left|L_{P^{\beta}_{n,r}}(f;y) - L_{P^{\beta}_{n,r}}(f;x)\right| &\le L_{P^{\beta}_{n,r}}(e_1, y-x)\\ &= M(y-x)\left(\tfrac{n-r}{n}A_{n-r} + \tfrac{r}{n}\right)\\ &\le M(y-x),\end{aligned}$$

since $(A_n) \to 1$ (see [3]), as it has been mentioned in the first section.

Case $0 < \alpha < 1$**.** We will apply the Hölder inequality for $p := \frac{1}{\alpha}$ and $q := \frac{1}{1-\alpha}$ s.t. $\frac{1}{p} + \frac{1}{q} = 1$. Taking into account that $L_{P^{\beta}_{n,r}}(e_0;x) = 1$, it results

$$\begin{aligned}&\left|L_{P^{\beta}_{n,r}}(f;y) - L_{P^{\beta}_{n,r}}(f;x)\right| \le\\ &\le M\left\{\sum_{l=0}^{n-r} p_{n-r,l}(y-x)\left[(1-(y-x))\tfrac{l}{n} + (y-x)\tfrac{l+r}{n}\right]\right\}^{\alpha} \cdot \left\{\sum_{l=0}^{n-r} p_{n-r,l}(y-x)\right\}^{1-\alpha}\\ &= M\left(L_{P^{\beta}_{n,r}}(e_1;y-x)\right)^{\alpha} \cdot \left(L_{P^{\beta}_{n,r}}(e_0;y-x)\right)^{1-\alpha}\\ &= M \cdot (y-x)^{\alpha} \cdot \left(\tfrac{n-r}{n}A_{n-r} + \tfrac{r}{n}\right)^{\alpha}\\ &\le M \cdot (y-x)^{\alpha}.\end{aligned}$$

□

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