JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY
J. Numer. Anal. Approx. Theory, vol. 54 (2025) no. 1, pp. 140–156, doi.org/10.33993/jnaat541-1397
ictp.acad.ro/jnaat

CONVERGENCE ANALYSIS OF ITERATIVE COMPOSITIONS IN NONLINEAR MODELING: EXPLORING SEMILOCAL AND LOCAL CONVERGENCE PHENOMENA

SUNIL KUMAR*, JANAK RAJ SHARMA† and IOANNIS K. ARGYROS‡

Abstract. In this work, a comprehensive analysis of a multi-step iterative composition for nonlinear equations is performed, providing insights into both local and semilocal convergence properties. At each step three linear systems are solved in the method, but with the same linear operator. The analysis covers a wide range of applications, elucidating the parameters affecting both local and semilocal convergence and offering insightful information for optimizing iterative approaches in nonlinear model-solving tasks. Moreover, we assert the solution's uniqueness by supplying the necessary standards inside the designated field. Lastly, we apply our theoretical deductions to real-world problems and show the related test results to validate our findings.

MSC. 65Y20, 65H10, 47H17, 41A58.
Keywords. Newton-type method, radius of convergence, Banach space, convergence, convergence order.

1. INTRODUCTION

The challenges inherent in exploring systems of nonlinear equations within the field of applied mathematics exhibit a remarkable diversity. While the specific methods for attaining analytical solutions vary depending on the problem, iterative approaches [3,11–13,18] commonly find utility in approximating solutions across a wide spectrum of problems. Under some standard assumptions, a typical representation for a nonlinear system takes the mathematical form:

$$F(x) = 0, \tag{1}$$

where $F : B_0 \subset B \to B_1$, B, B_1 are Banach spaces, and B_0 is an open convex set.

*Department of Mathematics, University Centre for Research and Development, Chandigarh University, Mohali-140413, India, e-mail: sfageria1988@gmail.com.
†Department of Mathematics, Sant Longowal Institute of Engineering and Technology, Longowal 148106, Punjab, India, e-mail: jrshira@yahoo.co.in.
‡Department of Computing and Mathematical Sciences, Cameron University, Lawton, OK 73505, USA, e-mail: iargyros@cameron.edu.

One of the fundamental one-point methods is Newton's method, which has quadratic convergence and is stated as

$$y_n = x_n - F'(x_n)^{-1}F(x_n), \qquad n = 0, 1, 2, \dots,$$

where $x_0 \in B_0$ is the starting point and $F' : B_0 \to \mathfrak{L}(B, B_1)$ is the first Fŕechet derivative of F. Here, $\mathfrak{L}(B, B_1)$ denotes the set of bounded linear operators from B to B_1. Many improved iterative methods have been presented and their convergence properties tested in Banach spaces (see, *e.g.*, [1, 3, 4, 6, 9, 10, 13–17, 19, 20] and related references).

A method established in [7] that is defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} w_n &= x_n - F'(x_n)^{-1}F(x_n), \\ y_n &= w_n - F'(x_n)^{-1}F(w_n), \\ z_n &= 2w_n - y_n \\ \text{and} \qquad x_{n+1} &= w_n - F'(x_n)^{-1}\big(-3F(w_n) + 3F(y_n) + 2F(z_n)\big), \end{aligned} \tag{2}$$

has received significant attention in this paper. Notice that at each step in method (2) three linear systems are solved, but with the same linear operator. A favorable comparison of this method with several competing methods can be found in [7]. Its convergence order has been shown to be five by establishing the error equation

$$e_{n+1} = (6A_2A_3A_2 - 8A_3A_2^2 + 6A_2^2A_3 + 14A_2^4)e_n^5 + \mathcal{O}(e_n^6), \tag{3}$$

where $e_n = x_n - x^*$ and $A_i = \frac{1}{i!}F'(x^*)^{-1}F^{(i)}(x^*)$, $i = 2, 3, \dots$, using the approach of Taylor series expansion. But there are notable restrictions with this approach which limit the applicability of the method. The convergence order five is achieved in [7] for $B = B_1 = \mathbb{R}^m$, (m is a natural number), and by assuming the existence of derivatives up to order five which are not used in (2). These conditions restrict the utilization of (2) to operators that are many times differentiable. Thus, there are even scalar equations for which the convergence of (2) cannot be assured. But the method (2) converges. Let us look at an example. Define the function $F : [-1.2, 1.2] \to \mathbb{R}$ by $F(t) = \theta_1 t^2 \log t + \theta_2 t^5 + \theta_3 t^4$ if $t \neq 0$, and $F(t) = 0$, if $t = 0$, where $\theta_1 \neq 0$, and $\theta_2 + \theta_3 = 0$. It follows by these definitions that the numbers 0 and 1 belong to domain of F, and $F(1) = 0$. But the function $F^{(3)}$ is not continuous at $t = 0$. Hence, the results in [7] cannot assure that $\lim_{n\to\infty} x_n = 1$. But (2) converges to x^* if, *e.g.*, $\theta_1 = \theta_2, \theta_3 = -1$, and $x_0 = 1.15$. This motivational example indicates that the conditions in [7] can be weakened. Moreover, there exist other limitations under with the usage of Taylor series.

In view of above discussion, the main motivation is to achieve the goal with weaker hypotheses rather than relying on earlier strong conditions. In the pursuit of enhancing the convergence characteristics, the present study investigates comprehensively the local and semilocal convergence analyses of (2).

Local convergence: Local convergence specifically addresses the behavior of an iterative method in the immediate vicinity of a solution. It explores the convergence properties within a small neighborhood around a solution point, providing a detailed analysis of how rapidly the iterative process refines its approximations when starting from nearby initial guesses. Understanding local convergence is important for assessing the robustness and effectiveness of an iterative algorithm in practical applications, where solutions are often sought in proximity to known or expected values.

Semilocal convergence: Semilocal convergence, on the other hand, refers to the behavior of an iterative method in a specific region of the solution space. Unlike global convergence, which considers convergence over the entire solution space, semilocal convergence focuses on the behavior of the iterative process within a limited neighborhood of a solution. It provides insights into how quickly the iterative scheme approaches a solution in a local region, offering valuable information about the convergence rate and efficiency near a specific point.

The rest of this article is organized as follows: the local convergence analysis is studied in Section 2, and the semilocal convergence analysis is studied in Section 3. Some special cases and applied problems are presented in Section 4 in order to further certify the theoretical deductions. In the end, the concluding remarks are added in Section 5.

2. CONVERGENCE 1: LOCAL

We introduce some scalar functions that play an important role in the local analysis of convergence for the method (2). Set $A = [0, +\infty)$.

Suppose :

(T_1) There exists a function $\varphi_0 : A \to A$ which is continuous as well as nondecreasing (FCND) on the interval A such that the equation $\varphi_0(t) - 1 = 0$ admits a smallest positive solution (SPS) denoted by s_0. Set $A_0 = [0, s)$.

(T_2) There exists a FCND $\varphi : A_0 \to A$. Moreover, define functions with domain A_0 and range $\mathbb{R}^+$ in turn by

$$h_1(t) = \frac{\int_0^1 \varphi((1-\theta)t)d\theta}{1-\varphi_0(t)},$$

$$\alpha(t) = \begin{cases} \varphi((1+h_1(t))t) \\ \varphi_0(t) + \varphi_0(h_1(t)t), \end{cases}$$

$$h_2(t) = \left[\frac{\int_0^1 \varphi((1-\theta)h_1(t)t)d\theta}{1-\varphi_0(h_1(t)t)} + \frac{\alpha(t)(1+\int_0^1 \varphi_0(\theta h_1(t)t)d\theta)}{(1-\varphi_0(t))(1-\varphi_0(h_1(t)t))}\right] h_1(t),$$

$$h_3(t) = \left[\frac{\int_0^1 \varphi((1-\theta)h_1(t)t)d\theta}{1-\varphi_0(h_1(t)t)} + \frac{\alpha(t)(1+\int_0^1 \varphi_0(\theta h_1(t)t)d\theta)}{(1-\varphi_0(t))(1-\varphi_0(h_1(t)t))}\right.$$

$$+ \frac{2(1+\int_0^1 \varphi_0(\theta h_1(t)t)d\theta)}{(1-\varphi_0(t))}\Big] h_1(t),$$

$$\beta(t) = 5\Big(1+\int_0^1 \varphi_0(\theta h_1(t)t)d\theta\Big)h_1(t) + 3\Big(1+\int_0^1 \varphi_0(\theta h_2(t)t)d\theta\Big)h_2(t)$$
$$+ 2\Big(1+\int_0^1 \varphi_0(\theta h_3(t)t)d\theta\Big)h_3(t),$$

and

$$h_4(t) = \frac{\int_0^1 \varphi((1-\theta)h_1(t)t)d\theta h_1(t)}{1-\varphi_0(h_1(t)t)} + \frac{\alpha(t)(1+\int_0^1 \varphi_0(\theta h_1(t)t)d\theta)h_1(t)}{(1-\varphi_0(t))(1-\varphi_0(h_1(t)t))} + \frac{\beta(t)}{(1-\varphi_0(t))}.$$

(T_3) The equation $h_j(t) - 1 = 0$ admits SPS in the interval A_0 denoted by δ_j, respectively. Define the parameter δ as

(4) $$\delta = \min\{\delta_j\}.$$

This parameter is shown to be a possible radius of convergence for the method (2) in Theorem 2.

The functions φ_0 and φ relate to the operators on the method.

(T_4) There exist an invertible operator E and a solution $x^* \in \Omega$ such that

$$\|E^{-1}(F'(x) - E)\| \le \varphi_0(\|x - x^*\|) \quad \text{for each } x \in \Omega.$$

Define the domain in $D = \Omega \bigcap M(x^*, s)$.

(T_5) $\|E^{-1}(F'(y) - F'(x))\| \le \varphi(\|y - x\|)$ for each $x, y \in D$ and

(T_6) $M[x^*, \delta] \subset \Omega$.

The conditions $(T_1) - (T_6)$ are employed to show the local analysis of convergence for the method (2).

Remark 1. *A usual choice for $E = I$ the identity operator or $E = F'(\bar{x})$ for $x \in \Omega$ an auxiliary point other than x^* or $E = F'(x^*)$. In the latter case according to the condition (T_3) the solution x^* is simple. However, this is not necessary the most flexible choice. Our approach proves the convergence of the method (2) to x^* even if the solution x^* is not simple provided that $E \neq F'(x^*)$ and the equation has only one solution in Ω.*

Next, the local analysis of convergence is established under the conditions $(T_1) - (T_6)$.

Theorem 2. *Suppose that the conditions $(T_1) - (T_6)$ hold and pick $x_0 \in M(x^*, \delta) - \{x^*\}$. Then, the sequence $\{x_n\}$ generated by the method (2) is well defined in the ball $M(x^*, \delta)$, remains in $M(x^*, \delta)$ for each $n = 0, 1, 2, \dots,$ and is convergent to x^*. Moreover, the following error estimates hold for each $n = 0, 1, 2, \dots,$*

(5) $$\|w_n - x^*\| \le h_1(\|\omega_n\|)\|\omega_n\| \le \|\omega_n\| < \delta,$$

(6) $$\|y_n - x^*\| \le h_2(\|\omega_n\|)\|\omega_n\| \le \|\omega_n\|,$$

(7) $$\|z_n - x^*\| \le h_3(\|\omega_n\|)\|\omega_n\| \le \|\omega_n\|,$$

$$\|x_{n+1}-x^*\| \leq h_4(\|\omega_n\|)\|\omega_n\| \leq \|\omega_n\|, \tag{8}$$

where, $\omega_n = x_n - x^*$*, the functions* h_j *are as previously defined and the radius* δ *is given by the formula* (4)*.*

Proof. Assertions (5)–(8) are shown by induction. Pick $v \in M(x^*, \delta)-\{x^*\}$. The application of the conditions (T_4) and (4) gives in turn

$$\|E^{-1}(F'(v)-E)\| \ \leq \varphi_0(\|v-x^*\|) \leq \varphi(\delta) < 1. \tag{9}$$

It follows by (9), and the Banach standard Lemma on linear operators [2] having inverses that $F'(v) \in \mathfrak{L}(B, B_0))$ as well as

$$\|F'(v)^{-1}E\| \leq \frac{1}{1-\varphi_0(\|v-x^*\|)}. \tag{10}$$

If $v = x_0$, the iterates w_0, y_0, z_0, and x_1 are well defined by the four substeps of the method (2), respectively. We shall also show that they belong in the ball $M(x^*, \delta)$ in turn as follows:

$$\begin{aligned} w_0 - x^* \ &= \omega_0 - F'(x_0)^{-1}F(x_0) \\ &= \int_0^1 F'(x_0)^{-1}(F'(x^*+\theta(\omega_0)) - F'(x_0))d\theta(\omega_0). \end{aligned} \tag{11}$$

Using (4), (10), estimate (10) (for $v = x_0$), and the definition of the function h_1, we have in turn

$$\begin{aligned} \|w_0 - x^*\| &\leq \frac{\int_0^1 \varphi((1-\theta)\|\omega_0\|)d\theta\|\omega_0\|}{1-\varphi_0(\|\omega_0\|)} \\ &\leq h_1(\|\omega_0\|)\|\omega_0\| \leq \|\omega_0\| < \delta, \end{aligned} \tag{12}$$

so the iterate $w_0 \in M(x^*, \delta)$, and the assertion (5) holds if $n = 0$.

We need the estimates

$$F(w_0) = F(w_0) - F(x^*) = \int_0^1 F'(x^*+\theta(w_0-x^*))d\theta(w_0-x^*). \tag{13}$$

Hence, by the condition (T_4)

$$\begin{aligned} \|E^{-1}F(w_0)\| &= \|E^{-1}\Big(\int_0^1 F'(x^*+\theta(w_0-x^*))d\theta - E + E\Big)(w_0-x^*)\| \\ &\leq \Big(1+\int_0^1 \varphi_0(\theta\|w_0-x^*\|)d\theta\Big)\|w_0-x^*\|, \\ F'(w_0)-F'(x_0) &= (F'(w_0)-F'(x^*)) + (F'(x^*)-F'(x_0)), \end{aligned} \tag{14}$$

so

(15)

$$\begin{aligned} \|E^{-1}(F'(w_0)-F'(x_0))\| \ &\leq \|E^{-1}(F'(w_0)-F'(x^*))\| + \|E^{-1}(F'(x_0)-F'(x^*))\| \\ &\leq \varphi_0(\|w_0-x^*\|) + \varphi_0(\|\omega_0\|) \\ &\leq \varphi_0(h_1(\|\omega_0\|)\|\omega_0\|) + \varphi_0(\|\omega_0\|) \\ &\leq \alpha(\|\omega_0\|) = \alpha_0 \end{aligned}$$

or

$$\|E^{-1}(F'(w_0)-F'(x_0))\| \leq \varphi(\|w_0-x_0\|) \leq \varphi(\|w_0-x^*\|+\|\omega_0\|) \leq \varphi((1+h_1(\|\omega_0\|))\|\omega_0\|) \leq \alpha(\|\omega_0\|). \tag{16}$$

Then, we can write from the second substep of the method (2) in turn that

$$y_0-x^* = w_0-x^*-F'(w_0)^{-1}F(w_0)+F'(w_0)^{-1}(F'(x_0)-F'(w_0))F'(x_0)^{-1}F(w_0). \tag{17}$$

By using (10) (for $v=x_0$), (12)–(17), the condition (T_5) and (4), we get in turn that

$$\begin{aligned}\|y_0-x^*\| &\leq \\ &\leq \Big[\frac{\int_0^1 \varphi((1-\theta)\|w_0-x^*\|)d\theta}{1-\varphi_0(\|w_0-x^*\|)} + \frac{\alpha_0(1+\int_0^1 \varphi_0(\theta\|w_0-x^*\|)d\theta)}{(1-\varphi_0(\|\omega_0\|))(1-\varphi_0(\|w_0-x^*\|))}\Big]\|w_0-x^*\| \\ &\leq\ h_2(\|\omega_0\|)\|\omega_0\| \leq \|\omega_0\|.\end{aligned}$$

Thus, the iterate $y_0 \in M(x^*,\delta)$ and the assertion (6) holds if $n=0$. Similarly, by the third substep of the method (2)

$$\begin{aligned} z_0-x^* &= w_0-x^*-F'(w_0)^{-1}F(w_0)+(F'(w_0)^{-1}+F'(x_0)^{-1})F(w_0) \\ &= w_0-x^*-F'(w_0)^{-1}F(w_0)+F'(w_0)^{-1}(F'(x_0)-F'(w_0))F'(x_0)^{-1}F(w_0) \\ &\quad +2F'(x_0)^{-1}F(w_0), \end{aligned} \tag{18}$$

leading to

$$\begin{aligned}\|z_0-x^*\| &\leq \Big[\frac{\int_0^1 \varphi((1-\theta)\|w_0-x^*\|)d\theta}{1-\varphi_0(\|w_0-x^*\|)} + \frac{\alpha_0(1+\int_0^1 \varphi_0(\theta\|w_0-x^*\|)d\theta)}{(1-\varphi_0(\|\omega_0\|))(1-\varphi_0(\|w_0-x^*\|))} \\ &\quad + \frac{2(1+\int_0^1 \varphi_0(\theta\|w_0-x^*\|)d\theta)}{(1-\varphi_0(\|\omega_0\|))}\Big]\|w_0-x^*\| \\ &\leq h_3(\|\omega_0\|)\|\omega_0\| \leq \|\omega_0\|.\end{aligned} \tag{19}$$

Hence, the iterate $z_0 \in M(x^*,\delta)$, and the assertion (7) holds if $n=0$. Moreover, by the last substep of the method (2), we have

$$\begin{aligned} x_1-x^* &= w_0-x^*-F'(w_0)^{-1}F(w_0)+F'(w_0)^{-1}(F'(x_0)-F'(w_0))F'(x_0)^{-1}F(w_0) \\ &\quad +5F'(x_0)^{-1}F(w_0)-3F'(x_0)^{-1}F(y_0)-2F'(x_0)^{-1}F(w_0), \end{aligned} \tag{20}$$

therefore

$$\|x_1-x^*\| \leq \frac{\int_0^1 \varphi((1-\theta)\|w_0-x^*\|)d\theta\|w_0-x^*\|}{1-\varphi_0(\|w_0-x^*\|)} \tag{21}$$

$$+\frac{\alpha_0(1+\int_0^1 \varphi_0(\theta\|w_0-x^*\|)d\theta)\|w_0-x^*\|}{(1-\varphi_0(\|\omega_0\|))(1-\varphi_0(\|w_0-x^*\|))}$$
$$+\frac{\beta_0}{(1-\varphi_0(\|\omega_0\|))} \leq h_4(\|\omega_0\|)\|\omega_0\| \leq \|\omega_0\|,$$

which shows the assertions (5)–(8) for $n = 0$, and $x_0, w_0, y_0, z_0, x_1 \in M(x^*, \delta)$. But these calculations can be repeated provided we replace x_0, w_0, y_0, z_0, x_1 by $x_m, w_m, y_m, z_m, x_{m+1}$ (m a natural number), respectively. Thus, the induction is completed, and $x_m, w_m, y_m, z_m, x_{m+1} \in M(x^*, \delta)$ for each $m = 0, 1, 2, \ldots$.

Furthermore, it follows from the estimation

$$\|\omega_{m+1}\| \leq c\|\omega_m\| < \delta, \tag{22}$$

where $c = h_4(\|\omega_0\|) \in [0, 1)$, that $\lim_{m\to\infty} x_m = x^*$ as well as $x_{m+1} \in M(x^*, \delta)$. □

In the next result we determine a domain that contains only x^* as a solution.

Proposition 3. *Suppose: the condition* (T_4) *holds on the ball* $M(x^*, s_1)$ *for some* $s_1 > 0$, *and there exists* $s_2 > s_1$ *such that*

$$\int_0^1 \varphi_0(\theta s_2)d\theta < 1. \tag{23}$$

Define the domain $D_1 = \Omega \cap M[x^*, s_2]$. *Then, the equation* $F(x) = 0$ *is uniquely solvable by* x^* *in the domain* D_1.

Proof. Let us assume that there exists $\bar{x} \in D_1$ solving the equation $F(x) = 0$. Define the linear operator $Q = \int_0^1 F'(x^* + \theta(\bar{x} - x^*))d\theta$. Then, it follows by the condition (T_4), and (23) that

$$\|E^{-1}(Q - E)\| \leq \int_0^1 \varphi_0(\theta\|\bar{x} - x^*\|)d\theta \leq \int_0^1 \varphi_0(\theta s_2)d\theta < 1.$$

Hence, the linear operator $Q^{-1} \in \mathfrak{L}(B, B_0)$. Moreover, from the identity

$$\bar{x} - x^* = Q^{-1}(F(\bar{x}) - F(x^*)) = Q^{-1}(0) = 0,$$

we deduce $\bar{x} = x^*$. □

Remark 4. *Clearly, we can choose* $s_1 = \delta$ *in Proposition 3.*

3. CONVERGENCE 2: SEMI-LOCAL

The roll of x^* and the functions φ_0, φ are exchanged by x_0, and the functions ψ_0, ψ, respectively.

Suppose:

(H_1) There exists FCND $\psi_0 : A \to A$, such that the equation $\psi_0(t) - 1 = 0$ has a SPS denoted by s_1.

Set $A_1 = [0, s_1)$.

(H_2) There exists FCND $\psi : A_1 \to A$. Define the sequence $\{a_n\}$ for $a_0 = 0$, some $b_0 \geq 0$, and each $n = 0, 1, 2, \dots$ by

$$
\begin{aligned}
\varrho_n &= \int_0^1 \psi((1-\theta)(b_n - a_n))d\theta(b_n - a_n), \\
c_n &= b_n + \frac{\varrho_n}{1 - \psi_0(a_n)}, \\
d_n &= c_n + 2(c_n - b_n), \qquad (24)\\
p_n &= \int_0^1 \psi((1-\theta)(c_n - a_n))d\theta(c_n - a_n) + (1 + \psi_0(a_n))(c_n - b_n), \\
q_n &= \int_0^1 \psi((1-\theta)(d_n - a_n))d\theta(d_n - a_n) + (1 + \psi_0(a_n))(d_n - b_n), \\
a_{n+1} &= d_n + \frac{2\varrho_n + 3p_n + 2q_n}{1 - \psi_0(a_n)}, \\
\mu_{n+1} &= \int_0^1 \psi(\theta(a_{n+1} - a_n))d\theta(a_{n+1} - a_n) + (1 + \psi_0(a_n))(a_{n+1} - b_n),
\end{aligned}
$$

and

$$
b_{n+1} = a_{n+1} + \frac{\mu_{n+1}}{1 - \psi_0(a_{n+1})}.
$$

The sequence $\{a_n\}$ is shown to be majorizing for the method (2) in Theorem 6. But first we need a general convergence condition for it.

(H_3) There exists $s_0 \in [0, s_1)$ such that for each $n = 0, 1, 2, \dots$

$$
\psi_0(a_n) < 1, \qquad \text{and} \quad a_n \leq s_0.
$$

It follows by this condition and (24) that

$$
0 \leq a_n \leq b_n \leq c_n \leq d_n \leq a_{n+1} < s_0,
$$

and there exists $a^* \in [0, s_0]$ such that $\lim_{n\to+\infty} a_n = a^*$. Notice that a^* is the least upper bound of the sequence $\{a_n\}$ which is unique.

As in the local analysis, we connect the functions ψ_0 and ψ to be operators on the method (2).

(H_4) There exists an invertible operator E and a point $x_0 \in \Omega$ such that $\|E^{-1}(F'(x) - E)\| \leq \psi_0(\|x - x_0\|)$ for each $x \in \Omega$. Notice that for $x = x_0$, the definition of s_1 and this condition imply

$$
\|E^{-1}(F'(x_0) - E)\| \leq \psi_0(0) < 1.
$$

So, the linear operator $F'(x_0)^{-1} \in \mathfrak{L}(B_0, B)$. Hence, we can set $b_0 \geq \|F'(x_0)^{-1}F(x_0)\|$. Define the domain $D_2 = \Omega \cap M(x_0, s_1)$.

(H_5) $\|E^{-1}(F'(y) - F'(x))\| \leq \psi(\|y - x\|)$ for each $x, y \in D_2$.

and

(H_6) $M[x_0, a^*] \subset \Omega$.

Remark 5. *Similar remarks as in Remark 1 follow, and $E = F'(x_0)$ is a possible choice.*

In the next result, we develop the semi-local analysis of convergence for the method (2) under the conditions $(H_1) - (H_6)$.

THEOREM 6. *Suppose that the conditions $(H_1) - (H_6)$ hold. Then, the sequence $\{x_n\}$ generated by (2) is well defined in the ball $M(x_0, a^*)$ remains in $M(x_0, a^*)$ for each $n = 0, 1, 2, \ldots,$ and is convergent to a solution $x^* \in M[x_0, a^*]$ of the equation $F(x) = 0$ such that*

$$\|x_n - x^*\| \leq a^* - a_n. \tag{25}$$

Proof. The following claims are demonstrated using induction

$$\|w_n - x_n\| \leq b_n - a_n, \tag{26}$$

$$\|y_n - w_n\| \leq c_n - b_n, \tag{27}$$

$$\|z_n - y_n\| \leq d_n - c_n, \tag{28}$$

and

$$\|x_{n+1} - z_n\| \leq a_{n+1} - d_n. \tag{29}$$

The assertions (26)-(29) are shown using induction. By the condition (H_4), the definition of b_0, and the first substep of the method (24), we have $\|w_0 - x_0\| = \|F'(x_0)^{-1}F(x_0)\| \leq b_0 = b_0 - a_0 < a^*$. So, the iterate $w_0 \in M(x_0, a^*)$, and the assertion (26) holds in $n = 0$. Let $v \in M(x_0, a^*)$.
Then, the definition of s_1 and the condition (T_4) imply

$$\|E^{-1}(F'(v) - E)\| \leq \psi_0\|v - x_0\| < 1,$$

thus $F'(v)^{-1} \in \mathfrak{L}(B, B_0)$ and

$$\|F'(v)^{-1}L\| \leq \frac{1}{1 - \psi_0(\|v - x_0\|)}. \tag{30}$$

Notice that by the existence of $F'(x_0)^{-1}$ the iterates w_0, y_0, z_0 and x_1 are well defined by the four substeps of the method (2), respectively. Next, we need in turn the estimates

$$F(w_m) = F(w_m) - F(x_m) - F'(x_m)(w_m - x_m),$$

and by the conditions (H_5), (30) (if $v = x_m$)

$$\begin{aligned}
\|E^{-1}F(w_m)\| =& \left\| \int_0^1 E^{-1}(F'(x_m + \theta(w_m - x_m)) - F'(x_m))d\theta(w_m - x_m) \right\| \\
&\leq \int_0^1 v(\theta\|w_m - x_m\|)d\theta\|w_m - x_m\| = \bar{\varrho}_m \\
&\leq \int_0^1 v(\theta\|b_m - a_m\|)d\theta\|b_m - a_m\| = \varrho_m, \\
y_m - w_m =& - F'(x_m)^{-1}F'(w_m), \\
\|y_m - w_m\| \leq& \|F'(x_m)^{-1}E\|\|E^{-1}F'(w_m)\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\bar{\varrho}_m}{1-\psi_0(\|x_m-x_0\|)} \leq \frac{\varrho_m}{1-\psi_0(a_m)} = c_m - b_m,\\
\|y_m - x_0\| &\leq \|y_m - w_m\| + \|w_m - x_0\|\\
&\leq c_m - b_m + b_m - a_0 = c_m < a^*,\\
z_m - y_m &= 2F'(x_m)^{-1}F'(w_m) = -2(y_m - w_m),\\
\|z_m - y_m\| &\leq 2\|y_m - w_m\| \leq 2(c_m - b_m) = d_m - c_m,\\
\|z_m - x_0\| &\leq \|z_m - y_m\| + \|y_m - x_0\| \leq d_m - c_m + c_m - a_0 = d_m < a^*,\\
F(y_m) &= F(y_m) - F(x_m) - F'(x_m)(w_m - x_m)\\
&= F(y_m) - F(x_m) - F'(x_m)(y_m - x_m) + F'(x_m)(y_m - w_m),
\end{aligned}
$$

(31)

$$
\begin{aligned}
\|E^{-1}F(y_m)\| &\leq \| \int_0^1 E^{-1}(F'(x_m + \theta(y_m - x_m)\|) - F'(x_m))d\theta(y_m - x_m)\|\\
&\quad + \|E^{-1}(F'(x_m) - E + E)(y_m - x_m)\|\\
&\leq \left\| \int_0^1 \psi(\theta\|y_m - x_m\|)d\theta \right\| \|y_m - x_m\|\\
&\quad + (1 + \psi_0(\|x_m - x_0\|))\|y_m - w_m\|\\
&= \bar{p}_m \leq \int_0^1 \psi(\theta(c_m - a_m))d\theta(c_m - a_m)\\
&\quad + (1 + \psi_0(a_m))(c_m - b_m) = p_m.
\end{aligned}
$$

Similarly by exchanging y_m by z_m in the previous calculation, and using

$$
F(z_m) = F(z_m) - F(x_m) - F'(x_m)(z_m - x_m) + F'(x_m)(z_m - x_m),
$$

we get

$$
\begin{aligned}
\|E^{-1}F(z_m)\| &\leq \bar{q}_m \leq q_m,\\
x_{m+1} - z_m &= 2F'(x_m)^{-1}F'(w_m) - 3F'(x_m)^{-1}F'(y_m) - 2F'(x_m)^{-1}F'(z_m),\\
\|x_{m+1} - z_m\| &\leq 2\|F'(x_m)^{-1}E\|\|E^{-1}F'(w_m)\| + 3\|F'(x_m)^{-1}E\|\|E^{-1}F'(y_m)\|\\
&\quad + 2\|F'(x_m)^{-1}E\|\|E^{-1}F'(z_m)\|\\
&\leq \frac{2\bar{\varrho}_m + 3\bar{p}_m + 2\bar{q}_m}{1-\psi_0(\|x_m - x_0\|)} \leq \frac{2\varrho_m + 3p_m + 2q_m}{1-\psi_0(a_m)} = a_{m+1} - d_m,\\
\|t_0\| &\leq \|x_{m+1} - z_m\| + \|z_m - x_0\|\\
&\leq a_{m+1} - d_m + d_m - a_0 = a_{m+1} < a^*,\\
F(x_{m+1}) &= F(x_{m+1}) - F(x_m) - F'(x_m)(w_m - x_m)\\
&= F(x_{m+1}) - F(x_m) - F'(x_m)(t_m) + F'(x_m)(t_m)\\
&\quad - F'(x_m)(w_m - x_m),\\
&= F(x_{m+1}) - F(x_m) - F'(x_m)(t_m) + F'(x_m)(x_{m+1} - w_m)
\end{aligned}
$$

(32)

$$\begin{aligned}\|E^{-1}F(x_{m+1})\| \le & \left\| \int_0^1 E^{-1}(F'(x_m+\theta(t_m)\|) - F'(x_m))d\theta(t_m) \right\| \\ & \quad + \|E^{-1}(F'(x_m)-E+E)(t_m)\| \\ \le & \left\| \int_0^1 \psi(\theta\|t_m\|)d\theta\|t_m\| + (1+\psi_0(\|x_m-x_0\|))\|x_{m+1}-w_m \right\| \\ = & \bar{\mu}_{m+1} \\ \le & \int_0^1 \psi(\theta(a_{m+1}-a_m))d\theta(a_{m+1}-a_m) \\ & \quad + (1+\psi_0(a_m))(a_{m+1}-b_m) = \mu_{m+1},\end{aligned}$$

(33)

$$\begin{aligned}\|w_{m+1}-x_{m+1}\| \ \le & \ \|F'(x_{m+1})^{-1}E\|\|E^{-1}F'(x_{m+1})\| \\ \le & \frac{\bar{\mu}_{m+1}}{1-\psi_0(\|t_0\|)} \le \frac{\mu_{m+1}}{1-\psi_0(a_{m+1})} = b_{m+1}-a_{m+1}, \\ \|w_{m+1}-x_0\| \ \le & \ \|w_{m+1}-x_{m+1}\| + \|t_0\| \\ \le & b_{m+1}-a_{m+1}+a_{m+1}-a_0 = b_{m+1} < a^*,\end{aligned}$$

where $t_m = x_{m+1} - x_m$.

Hence, the assertions (26)–(29) hold for each $m = 0, 1, 2, \ldots$, and all iterates x_m, w_m, y_m, z_m belong in $M(x_0, a^*)$. Moreover, it follows by the condition (H_3) that the sequence $\{a_m\}$ is Cauchy as convergent. Then, by the triangle inequality and (26)–(29)

$$\begin{aligned}\|t_m\| \ \le & \ \|x_{m+1}-z_m\| + \|z_m-y_m\| + \|y_m-w_m\| + \|w_m-x_m\| \\ \le & \ a_{m+1}-d_m+d_m-c_m+c_m-b_m+b_m-a_m = a_{m+1}-a_m,\end{aligned}$$

i.e.,

(34) $$\|t_m\| \ \le a_{m+1}-a_m.$$

Hence, the sequence $\{a_m\}$ is also Cauchy in the Banach space B_0, and consequently, there exists $x^* \in M[x_0, a^*]$ such that $\lim_{m\to+\infty} x_m = x^*$. Furthermore, using the continuity of operator F, and by letting $m \to +\infty$ in the estimate (32) we deduce that $F(x^*) = 0$. Let $k = 0, 1, 2, \ldots$. Then, by (34), we obtain

(35) $$\|t_m\| \le a_{m+k} - a_m.$$

Finally, by letting $k \to +\infty$ in (35), we get the assertion (25). □

The uniqueness ball is determined in the next result.

PROPOSITION 7. *Suppose: There exists a solution $\bar{x} \in M(x_0, s_3)$ of the equation $F(x) = 0$ for some $s_3 > 0$. The condition (H_4) holds on the ball*

$M(x_0, s_3)$ and there exists $s_4 \geq s_3$ such that

$$\int_0^1 \psi_0((1-\theta)s_3 + \theta s_4)d\theta < 1. \tag{36}$$

Define the domain $D_3 = \Omega \bigcap M[x_0, s_4]$. Then, the only solution of the equation $F(x) = 0$ in the domain D_3 is $\bar{x}$.

Proof. Let $u \in D_3$ be such that $F(u) = 0$. Define the linear operator $Q_1 = \int_0^1 F'(\bar{x} + \theta(u - \bar{x}))d\theta$. It follows by the condition (H_4), and (30) that

$$\begin{aligned} \|E^{-1}(Q_1 - E)\| &\leq \int_0^1 \psi_0((1-\theta)\|\bar{x} - x_0\| + \theta\|u - \bar{x}\|)d\theta \\ &\leq \int_0^1 \psi_0(\theta s_4 + (1-\theta)s_3)d\theta < 1. \end{aligned}$$

Then, the identity

$$u - \bar{x} = Q_1^{-1}(F(u) - F(\bar{x})) = Q^{-1}(0) = 0,$$

we conclude $u = \bar{x}$. □

REMARK 8. (i) *In the condition (H_6), the limit point a^* can be replaced by s_0.*

(ii) *In Proposition 7, we can set $\bar{x} = x^*$, and $s_3 = a^*$ under all the conditions of Theorem 6.*

4. NUMERICAL RESULTS

The numerical tests contribute to a deeper understanding of the convergence properties of iterative compositions, enhancing the practical applicability and theoretical foundation of nonlinear modeling techniques. In view of this, here we verify the theoretical results proven in the preceding sections. Let us consider the following problems:

EXAMPLE 9. *Consider the equation*

$$F(x) = \ x - \beta sin(x) - K = 0, \tag{37}$$

where $0 \leq \beta < 1$, $0 \leq K \leq \pi$, that comes from Kepler's [5]. In [5], several options are provided for the values of β and K. Specifically, the approximate solution to (37) is $x^* \approx 0.13320215\ldots$ for $K = 0.1$ and $\beta = 0.25$. Let $D = S(x^*, c)$ be the initial approximation such that $x^{(0)} = \frac{3}{4} \in D$, with c being a positive constant. Now, we have

$$F'(x) = \ 1 - \beta cos(x).$$

Thus, $\forall\ x, y \in D$, we get the approximation,

$$\begin{aligned} |F'(x^*)^{-1}(F'(x) - F'(y))| &= \frac{|\beta(\cos(x) - \cos(y))|}{|1 - \beta\cos(x^*)|} \\ &= \frac{2|\beta| \cdot |\sin(\frac{x+y}{2})\sin(\frac{x-y}{2})|}{|1 - \beta\cos(x^*)|} \end{aligned}$$

$$\leq L_0|x-y|,$$

and

$$|F'(x^{(0)})^{-1}(F'(x)-F'(y))| \leq L_1|x-y|,$$

where $L_0 = \frac{|\beta|}{|1-\beta\cos(x^*)|} \approx 0.332352$ *and* $L_1 = \frac{|\beta|}{|1-\beta\cos(x^{(0)})|} \approx 0.305968.$

The aforesaid approximations lead to the estimation of parameters utilised in the conditions of Section 2 and Section 3. The parameters listed in $(T_1)-(T_4)$ *are given as*

$$\varphi_0(t) = L_0 t, \qquad \varphi(t) = L_0 t,$$

and

$$\delta = \min\{2.00591, 1.45636, 1.01559, 0.413765\} = 0.413765.$$

Moreover, the parameters defined in $(H_1)-(H_5)$ *are chosen as*

$$\psi_0(t) = L_0 t, \qquad \psi(t) = L_0 t, \qquad b_0 = 0.0122$$

and consequently, we obtain the sequence $\{a_n\}$ *as*

$$\{a_n\}_{n\geq 1} = \{0.0126708\ldots, 0.0131713\ldots, 0.0131721\ldots, \cdots\cdots\},$$

which converges to $a_* \approx 0.0132 < s_1 = 3.00886.$

EXAMPLE 10. *The norm* $\|x\| = \max_{1\leq i\leq m}|x_i|$ *for every* $x = (x_1, x_2, \ldots, x_m)^T \in \mathbb{R}^m$ *and matrix norm* $\|A\| = \max_{1\leq i\leq m}\sum_{j=1}^{j=m}|a_{ij}|$ *for any* $A = (a_{ij})_{1\leq i,j\leq m} \in \mathfrak{L}(\mathbb{R}^m)$. *We can take the domain* $\mathbb{R}^m$, *for every* $m \geq 2$.

On a closed interval $[0,1]$, *define the boundary value problem as*

$$x''(t) = -x(t)^2, \quad x(0) = \quad x(1) = 0. \tag{38}$$

Taking into consideration the partitioning of $[0,1]$ *with a sub-interval of length* $h = 1/k$ *as*

$$t_0 = 0 < t_1 < t_2 < \cdots < t_{k-1} < t_k = 1$$

in order to convert the equation (38) *into a finite dimensional problem.*

Denoting $x_i = x(t_i)\ \forall\ i$, *and by finite differences*

$$x_i'' \approx \frac{x_{i+1} - 2x_i + x_{i-1}}{h^2},$$

$\forall\ i = 1, 2, \ldots, k-1$, *equation* (38) *reduces into nonlinear system,* $F : D \subseteq \mathbb{R}^{k-1} \to \mathbb{R}^{k-1}$, *given by*

$$x_{i+1} - 2x_i + h^2x_i^2 + x_{i-1} = 0, \quad i = 1, 2, 3, \ldots, k-1, \tag{39}$$

where $x_0 = 0 = x_k$. *Now at* $x = (x_1, x_2, \ldots, x_{k-1})^T \in D$ *the Fŕechet derivative is given as follows:*

$$F'(x) = \begin{pmatrix} 2h^2x_1 - 2 & 1 & 0 & \ldots & 0 \\ 1 & 2h^2x_2 - 2 & 1 & \ldots & 0 \\ 0 & 1 & 2h^2x_3 - 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2h^2x_{k-1} - 2 \end{pmatrix}.$$

In specifically, we select $k = 101$ *to find the parameters provided in the* Section 2 *and* Section 3 *to convert* (39) *to a system of* 100 *equations fulfilling the solution* $x^* = (0, \overset{100}{\cdots}, 0)^T$.

Furthermore, we choose the initial estimate as $x^0 = (0.5, \overset{100}{\cdots}, 0.5)^T \in D$, *treating the domain* $D = S(x^*, c)$ *as an open ball for some positive constant* c. *Then, we can determine that*

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L_0\|x - y\|,$$

and

$$\|F'(x^0)^{-1}(F'(x) - F'(y))\| \leq L_1\|x - y\|,$$

where $L_0 = 0.24999$ *and* $L_1 = 0.27896$, *for any* $x, y \in D$.

The parameters listed in Section 2 *under* $(T_1) - (T_4)$ *conditions for the local convergence analysis are selected as follows in view of the aforementioned approximations:*

$$\varphi_0(t) = L_0 t, \qquad \varphi(t) = L_0 t,$$

and consequently, we have that

$$\delta = \min\{2.666773, 1.936172, 1.350181, 0.550085\} = 0.550085.$$

Additionally, for the semilocal convergence analysis, the parameters defined in Section 3 *under conditions* $(H_1) - (H_5)$ *are selected as*

$$\psi_0(t) = L_0 t, \qquad \psi(t) = L_0 t, \qquad b_0 = 0.025$$

and consequently, we have the sequence $\{a_n\}$ *as*

$$\{a_n\}_{n \geq 1} = \{0.0264888\ldots, 0.0280819\ldots, 0.0280883\ldots, \cdots\cdots\},$$

which converges to $a_* \approx 0.0281 < s_1 = 4$. *These results confirm* Section 2 *and* Section 3 *conditions.*

EXAMPLE 11. *Let* $C[0,1]$ *stand for the continuous function space with norm* $\|x\| = sup_{0 \leq t \leq 1}|x(t)|$ *for each* $x \in C[0,1]$ *and defined on the domain as a closed unit interval* $[0,1]$. *Let* $D = \{x \in C[0,1], \|x\| < 1\}$ *and nonlinear mapping (see [8])* $F : D \to C[0,1]$ *as*

$$F(x)(t) = x(t) - \mu \int_0^1 k(s,t)x(s)^3 ds, \quad t \in [0,1], \quad x \in D, \tag{40}$$

where $\mu \in \mathbb{R}$, *and the kernel* $k(s,t)$ *is given as*

$$k(s,t) = \begin{cases} (1-s)t, & t \leq s, \\ s(1-t), & s \leq t, \end{cases}$$

that satisfies the following,

$$\left\| \int_0^1 k(s,t) ds \right\| \leq \tfrac{1}{8}.$$

Moreover, the Fréchet derivative of (40) is given by

$$F'(x)\kappa(t) = \kappa(t) - 3\mu \int_0^1 k(s,t)x(s)^2\kappa(s)ds, \qquad \kappa \in D.$$

Note that solution of (40) is $x^* = 0$ and also satisfies $F'(x^*) = I$. Then, for $x, y \in D$, we have,

$$\begin{aligned}\|F'(x^*)^{-1}(F'(x) - F'(y))\| &\leq 3|\mu| \Big\| \int_0^1 k(s,t)(x(s)^2 - y(s)^2)\kappa(s)ds \Big\| \\ &\leq L_0\|x - y\|,\end{aligned}$$

where $L_0 = \frac{3|\mu|}{4}$.

Furthermore, for the $x^{(0)} \in D$ which is given as $x^{(0)}(t) = \frac{1}{2}$, $t \in [0,1]$, and the estimation

$$\|I - F'(x^{(0)}\| \leq 3|\mu| \Big\| \int_0^1 k(s,t)x^{(0)}(s)^2\kappa(s)ds \Big\| \leq \tfrac{3|\mu|}{32},$$

it is calculated that $F'(x^{(0)})^{-1} \leq \frac{32}{32-3|\mu|}$, provided $|\mu| < \frac{32}{3}$. Therefore, $\forall$ $x, y \in D$, we get

$$\|F'(x^{(0)})^{-1}(F'(x) - F'(y))\| \leq L_1\|x - y\|,$$

and

$$\|F'(x^{(0)})^{-1}F'(x^{(0)})\| \leq L_2,$$

where $L_1 = \frac{24|\mu|}{32-3|\mu|}$ and $L_2 = (1 + \frac{|\mu|}{32})\frac{16}{32-3|\mu|}$.

We particularly fix $\mu = \frac{1}{2}$, in the above approximations, for parameters listed in Section 2 and Section 3. The parameters used in the conditions $(T_1) - (T_4)$ are defined as

$$\varphi_0(t) = L_0t, \qquad \varphi(t) = L_0t,$$

and so

$$\delta = \min\{1.77778, 1.29073, 0.900085, 0.366709\} = 0.366709.$$

Furthermore, the parameters defined in $(H_1) - (H_5)$ are chosen as

$$\psi_0(t) = L_0t, \qquad \psi(t) = L_0t, \qquad b_0 = 0.0156$$

and consequently, we obtain the sequence $\{a_n\}$ as

$$\{a_n\}_{n\geq 1} = \{0.0164694\ldots, 0.0173984\ldots, 0.0174017\ldots, \cdots\cdots\},$$

which converges to $a_* \approx 0.0174 < s_1 = 2.66667$. These results confirm Section 2 and Section 3 conditions.

5. CONCLUSION

Comprehensive analysis is conducted on a fifth-order iterative technique to assess its local and semilocal convergence in Banach Spaces. In contrast to the conventional reliance on Taylor series expansions, this study establishes generalized convergence results based solely on assumptions about first-order derivatives. The presented analysis introduces a fresh perspective for examining the convergence of the iterative method, focusing exclusively on the operators inherent in the given iterative processes. Unlike earlier studies, which incorporated higher-order derivatives not present in the methods under consideration, this approach acknowledges the potential non-existence of such derivatives. Consequently, previous results do not provide a definitive guarantee of convergence, even though it may occur. This innovative approach effectively broadens the applicability of the given method to a more extensive range of problems. Rigorous testing on applied problems lends support to the validity of the developed results. A noteworthy observation is that the analytical technique employed in this study has broader applicability and could be extended to enhance the effectiveness of other methods in a similar manner.

REFERENCES

[1] A.R. Amiri, A. Cordero, M.T. Darvishi, J.R. Torregrosa, *Preserving the order of convergence: Low-complexity Jacobian-free iterative schemes for solving nonlinear systems*, J. Comput. Appl. Math., **337** (2018), pp. 87–97. https://doi.org/10.1016/j.cam.2018.01.004

[2] I.K. Argyros, Á.A. Magreñán, *Iterative Methods and Their Dynamics with Applications*, CRC Press, New York, 2017.

[3] I.K. Argyros, *The Theory and Applications of Iteration Methods*, CRC Press, New York, 2022.

[4] R. Behl, I.K. Argyros, *Local convergence for multi-step high order solvers under weak conditions*, Mathematics, **8** (2020), no. 179. https://doi.org/10.3390/math8020179

[5] J.M.A. Danby, T.M. Burkardt, *The solution of Kepler's equation*, Celest. Mech., **31** (1983), pp. 95–107.

[6] M.T. Darvishi, A. Barati, *A third-order Newton type method to solve systems of nonlinear equations*, Appl. Math. Comput., **187** (2007), pp. 630–635. https://doi.org/10.1016/j.amc.2006.08.080

[7] R. Erfanifar, M. Hajarian, *A new multi-step method for solving nonlinear systems with high efficiency indices*, Numer. Algor., **97** (2024), pp. 959–984. https://doi.org/10.1007/s11075-023-01735-2

[8] J.M. Gutiérrez, Á.A. Magreñán, N. Romero, *On the semilocal convergence of Newton-Kantorovich method under center-Lipschitz conditions*, Appl. Math. Comput., **221** (2013), pp. 79–88. https://doi.org/10.1016/j.amc.2013.05.078

[9] H.H.H. Homeier, *On Newton-type methods with cubic convergence*, J. Comput. Appl. Math., **176** (2005), pp. 425–432. https://doi.org/10.1016/j.cam.2004.07.027

[10] D. Kumar, S. Kumar, J.R. Sharma, L. Jäntschi, *Convergence analysis and dynamical nature of an efficient iterative method in Banach spaces*, Mathematics, **9** (2021), no. 2510. https://doi.org/10.3390/math9192510

[11] Á.A. Magreñán, I.K. Argyros, *A contemporary study of iterative methods: convergence dynamics and applications*, Academic Press, Elsevier, 2019.

[12] J.M. Ortega, W.C. Rheinboldt, *Iterative Solutions of Nonlinear Equations in Several Variables*, Academic Press, New York, 1970.

[13] A.M. Ostrowski, *Solution of Equations and Systems of Equations*, Academic Press, New York, 1960.

[14] P.D. Proinov, *Semi-local convergence of two iterative methods for simultaneous computation of polynomial zeros*, C. R. Acad. Bulgare Sci., **59** (2006), pp. 705–712.

[15] W.C. Rheinboldt, *An adaptive continuation process for solving systems of nonlinear equations*, Banach Center Publ., **3** (1978), pp. 129–142.

[16] S.M. Shakhno, *Convergence of the two-step combined method and uniqueness of the solution of nonlinear operator equations*, J. Comput. Appl. Math., **261** (2014), pp. 378–386. https://doi.org/10.1016/j.cam.2013.11.018

[17] J.R. Sharma, S. Kumar, I.K. Argyros, *Generalized Kung-Traub method and its multi-step iteration in Banach spaces*, J. Complexity, **54** (2019), 101400. https://doi.org/10.1016/j.jco.2019.02.003

[18] J.F. Traub, *Iterative Methods for the Solution of Equations*, Chelsea Publishing Company, New York, 1970.

[19] X.Y. Xiao, H. Yin, *Achieving higher order of convergence for solving systems of nonlinear equations*, Appl. Math. Comput., **311** (2017), pp. 251–261. https://doi.org/10.1016/j.amc.2017.05.033

[20] X.Y. Xiao, H. Yin, *Accelerating the convergence speed of iterative methods for solving nonlinear systems*, Appl. Math. Comput., **333** (2018), pp. 8–19. https://doi.org/10.1016/j.amc.2018.03.108

Received by the editors: February 02, 2024; accepted: May 13, 2025; published online: June 30, 2025.