JOURNAL OF NUMERICAL ANALYSIS AND APPROXIMATION THEORY

J. Numer. Anal. Approx. Theory, vol. 54 (2025) no. 1, pp. 100–110, doi.org/10.33993/jnaat541-1515

ictp.acad.ro/jnaat

UPPER AND LOWER SOLUTION METHOD FOR CONTROL OF SECOND-ORDER KOLMOGOROV TYPE SYSTEMS

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Abstract. In this paper, an upper and lower solution method for the control of second-order Kolmogorov systems is introduced. Two iterative algorithms, one exact and one approximate, are proposed and their convergence is studied. The technique is based on Perov's fixed point theorem, matrices convergent to zero, and the use of Bielecki's norm.

MSC. 34H05, 37N25, 34A12, 34K35.

Keywords. Kolmogorov system, Lotka-Volterra system, control problem, fixed point, lower and upper solutions, approximation algorithm, Volterra-Fredholm integral equation, Bielecki norm.

1. INTRODUCTION AND PRELIMINARIES

In this paper we consider systems of second-order Kolmogorov type equations. In paper [5] we discussed a control problem related to first-order Kolmogorov systems, with full reference to the Lotka-Volterra system and the SIR model, with the controllability condition $\varphi(x, y) = 0$. Such kind of systems appear in several fields, such as population dynamics, ecological balance and medicine (see, for example, [1, 2, 9, 10, 17]). In paper [8], we introduced the Kolmogorov type second-order equations and using a fixed point approach we studied various control problems related to them (see also [3, 5, 6, 7, 8, 12]).

In this paper we deal with the control of second-order Kolmogorov type system,

$$\begin{cases} \left(\frac{x'(t)}{x(t)}\right)' = f(x(t), y(t), \lambda) \\ \left(\frac{y'(t)}{y(t)}\right)' = g(x(t), y(t), \lambda), \end{cases} \tag{1}$$

for $t \in [0, T]$, with the initial conditions

$$x(0) = a, \quad x'(0) = 0, \quad y(0) = b, \quad y'(0) = 0, \tag{2}$$

where $a, b > 0$. Here the control λ is a vector from $\mathbb{R}^m$, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. Such kind of problems have applications to various domains, particularly in

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biomathematics. The controllability condition is

$$\Psi\big(x(T), y(T)\big) = 0,$$

where $\Psi : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. For example, we can take

$$\Psi(s,\tau) = s - k\tau \quad \text{or} \quad \Psi(s,\tau) = s - k,$$

with k a given constant.

First, we introduce the notions of lower and upper solutions of the control problem (see [11], [5]).

Definition 1. *We call a lower solution of the control problem, a triple* $(\underline{x}, \underline{y}, \underline{\lambda})$ *where* $(\underline{x}, \underline{y})$ *is a solution of the Cauchy problem with* $\lambda = \underline{\lambda}$ *and*

$$\Psi\big(\underline{x}(T), \underline{y}(T)\big) < 0.$$

Definition 2. *A triple* $(\overline{x}, \overline{y}, \overline{\lambda})$ *is said to be an upper solution of the control problem if* $(\overline{x}, \overline{y})$ *is a solution of the Cauchy problem with* $\lambda = \overline{\lambda}$ *and*

$$\Psi\big(\overline{x}(T), \overline{y}(T)\big) > 0.$$

Lower and upper solutions can be obtained with the aid of the computer by repeated trials giving various control variable values.

The purpose of this paper is to present an algorithm for solving the above control problem. The convergence of the algorithm is proved. By this algorithm and an iterative method, the controllability of the problem is obtained.

We finish this section by some preliminary notions and results (see, *e.g.* [13, 14, 16]).

Theorem 3 (Perov). *Let* $(X, \|\cdot\|)$ *be a Banach space,* D *a closed subset of* $X \times X$ *and* $N : D \to D$, $N = (A, B)$, $A, B : D \to X$ *be an operator satisfying the following vector inequality*

$$\begin{bmatrix} \|A(x) - A(y)\| \\ \|B(x) - B(y)\| \end{bmatrix} \leq M \begin{bmatrix} \|x_1 - y_1\| \\ \|x_2 - y_2\| \end{bmatrix}$$

for all $x = (x_1, x_2)$, $y = (y_1, y_2) \in D$, *where* M *is a matrix of size two that is convergent to zero. Then* N *has a unique fixed point in* D *which is the limit of the sequence* $\left(N^k(x)\right)_{k \geq 1}$ *of successive approximations starting from any initial point* $x \in D$.

By a matrix that converges to zero we mean a square matrix M with nonnegative entries and the property that its power M^k converges to the zero matrix as $k \to \infty$. It is well-known (see [13]) that this property is equivalent to the fact that the spectral radius of M is strictly less than one, and also to the fact that the matrix $I - M$ (I being the unit matrix of the same size) is nonsingular and its inverse also has nonnegative entries. We mention that a square matrix of size two $M = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ with nonnegative entries is convergent to zero if and only if

$$\operatorname{tr} M < \min\{2,\ 1 + \det M\}, \tag{3}$$

that is

$$a + d < 2 \quad \text{and} \quad a + d < 1 + ad - bc. \tag{4}$$

Note that if M is convergent to zero, then $a < 1$ and $d < 1$.

When dealing with Volterra type integral equations it is convenient that instead of the max-norm $\|x\|_\infty = \max_{t\in[a,b]} |x(t)|$ on the space $C[a,b]$, to consider an equivalent norm defined by

$$\|x\|_\theta = \max_{t\in[a,b]} \left(|x(t)| \, e^{-\theta(t-a)} \right),$$

for some suitable number $\theta > 0$. Such a norm is called a *Bielecki norm* and it is equivalent to the max-norm, as follows from the inequalities

$$e^{-\theta(b-a)} \|x\| \le \|x\|_\theta \le \|x\| \qquad (x \in C[a,b]).$$

The trick of using Bielecki norms consists in the possibility to choose suitable large enough θ in order to make constants smaller in Lipschitz or growth conditions.

2. MAIN RESULTS

In order to give the algorithm, we need to guarantee that the Cauchy problem (1)–(2) has a unique solution for each λ and depends continuously on parameter λ.

THEOREM 4. *Let* $\alpha = \ln a$, $\beta = \ln b$ *and*

$$\rho \ge \exp\left(1 + \max\{|\alpha|, |\beta|\}\right). \tag{5}$$

Assume that $f, g : \mathbb{R}^2 \times \mathbb{R}^m \to \mathbb{R}$ *with* $f(0, y, \lambda) \equiv 0$, $g(x, 0, \lambda) \equiv 0$, *for any* $x, y \in \mathbb{R}$, $\lambda \in \mathbb{R}^m$ *satisfy the Lipschitz conditions*

$$\begin{aligned} |f(x,y,\lambda) - f(\overline{x},\overline{y},\mu)| &\le a_{11}|x-\overline{x}| + a_{12}|y-\overline{y}| + a_{13}|\lambda-\mu|, \\ |g(x,y,\lambda) - g(\overline{x},\overline{y},\mu)| &\le a_{21}|x-\overline{x}| + a_{22}|y-\overline{y}| + a_{23}|\lambda-\mu|, \end{aligned}$$

for all $x, y, \overline{x}, \overline{y} \in \mathbb{R}$, $\lambda, \mu \in \mathbb{R}^m$ *and some nonnegative numbers* a_{ij} $(i = 1,2;\ j = 1,2,3)$. *In addition, assume that the matrix*

$$M = \frac{\rho T^2}{2} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{6}$$

is convergent to zero. Then, for any $\lambda \in \mathbb{R}^m$, *the Cauchy problem* (1)–(2) *has a unique solution* (x, y) *satisfying* $\|x\|_\infty \le \rho$ *and* $\|y\|_\infty \le \rho$, *which depends continuously on the parameter* λ.

Proof. 1. Fixed point formulation of the Cauchy problem.
Making the change of variables $x = e^u$ and $y = e^v$ yields the system

$$\begin{cases} u''(t) = f\left(e^{u(t)}, e^{v(t)}, \lambda\right) \\ v''(t) = g\left(e^{u(t)}, e^{v(t)}, \lambda\right), \end{cases}$$

under the initial conditions $u\,(0)=\alpha,\ u'\,(0)=0,\ v\,(0)=\beta$ and $v'\,(0)=0$. Successive integrations lead to the integral system

$$\begin{cases} u(t)=\alpha+\int_0^t\int_0^\tau f(e^{u(s)},e^{v(s)},\lambda)dsd\tau \\ v(t)=\beta+\int_0^t\int_0^\tau g(e^{u(s)},e^{v(s)},\lambda)dsd\tau, \end{cases} \tag{7}$$

which can be seen as a fixed point equation for the operator $N=(A,B)$, where

$$A\,(u,v)\,(t)=\alpha+\int_0^t\int_0^\tau f(e^{u(s)},e^{v(s)},\lambda)dsd\tau,$$
$$B\,(u,v)\,(t)=\beta+\int_0^t\int_0^\tau g(e^{u(s)},e^{v(s)},\lambda)dsd\tau.$$

We shall apply Perov's fixed point theorem in the set

$$D_R:=\{(u,v)\in C([0,T];\mathbb{R}^2):\ \|u\|_\infty\le R,\ \|v\|_\infty\le R\},$$

where $R=\ln\rho$.

2. Operator N is a Perov contraction.

Let $(u,v),(\overline{u},\overline{v})\in D_R$. One has

$$\begin{aligned}|A(u,v)(t)-A(\overline{u},\overline{v})(t)| &\le \int_0^t\int_0^\tau \left|f(e^{u(s)},e^{v(s)},\lambda)-f(e^{\overline{u}(s)},e^{\overline{v}(s)},\lambda)\right|\,ds\,d\tau \\ &\le \int_0^T\int_0^\tau\left(a_{11}\left|e^{u(s)}-e^{\overline{u}(s)}\right|+a_{12}\left|e^{v(s)}-e^{\overline{v}(s)}\right|\right)ds\,d\tau.\end{aligned}$$

Furthermore, using the Lagrange mean value theorem we deduce that

$$\left|e^{u(s)}-e^{\overline{u}(s)}\right|\le\rho\,|u(s)-\overline{u}(s)|$$

and

$$\left|e^{v(s)}-e^{\overline{v}(s)}\right|\le\rho\,|v(s)-\overline{v}(s)|\,,$$

and consequently

$$\begin{aligned}&\int_0^T\int_0^\tau\left(a_{11}|e^{u(s)}-e^{\overline{u}(s)}|+a_{12}|e^{v(s)}-e^{\overline{v}(s)}|\right)ds\,d\tau\le \\ &\le\tfrac{\rho T^2}{2}\left(a_{11}\|u-\overline{u}\|_\infty+a_{12}\|v-\overline{v}\|_\infty\right).\end{aligned}$$

It follows that

$$\|A(u,v)-A(\overline{u},\overline{v})\|_\infty\le\tfrac{\rho T^2}{2}\Big(a_{11}\,\|u-\overline{u}\|_\infty+a_{12}\,\|v-\overline{v}\|_\infty\Big).$$

Similarly,

$$\|B(u,v)-B(\overline{u},\overline{v})\|_\infty\le\tfrac{\rho T^2}{2}\Big(a_{21}\,\|u-\overline{u}\|_\infty+a_{22}\,\|v-\overline{v}\|_\infty\Big).$$

These two inequalities can be written in the vector form

$$\begin{bmatrix}\|A(u,v)-A(\bar{u},\overline{v})\|_\infty\\ \|B(u,v)-B(\bar{u},\overline{v})\|_\infty\end{bmatrix}\le M\begin{bmatrix}\|u-\bar{u}\|_\infty\\ \|v-\overline{v}\|_\infty\end{bmatrix}.$$

Since matrix M is assumed to be convergent to zero, the operator $N = (A, B)$ is a Perov contraction on D_R.

3. Invariance of the set D_R.

We show that

$$\|u\|_\infty \leq R,\ \|v\|_\infty \leq R \quad \text{imply} \quad \|A(u,v)\|_\infty \leq R,\ \|B(u,v)\|_\infty \leq R.$$

First, note that

$$|A(u,v)(t)| \leq |\alpha| + \int_0^t \int_0^\tau \left| f(e^{u(s)}, e^{v(s)}, \lambda) \right| ds d\tau.$$

Since $f(0, y, \lambda) \equiv 0$, one has

$$\left| f\left(e^{u(s)}, e^{v(s)}, \lambda\right) \right| = \left| f\left(e^{u(s)}, e^{v(s)}, \lambda\right) - f\left(0, e^{v(s)}, \lambda\right) \right| \leq a_{11} e^{u(s)} \leq a_{11}\rho.$$

Given that the elements from the first diagonal of the matrix M are less than one, *i.e.*,

$$\tfrac{a_{11}\rho T^2}{2} < 1 \quad \text{and} \quad \tfrac{a_{22}\rho T^2}{2} < 1,$$

we have

$$|A(u,v)(t)| \leq |\alpha| + \tfrac{a_{11}\rho T^2}{2} < |\alpha| + 1.$$

Similarly

$$|B\,(u,v)\,(t)| \leq |\beta| + \tfrac{a_{22}\rho T^2}{2} < |\beta| + 1.$$

Thus, the set D_R is invariant by N provided

$$1 + \max\left\{|\alpha|, |\beta|\right\} \leq R,$$

which is true in virtue of condition (5).

Therefore, the operator $N = (A, B)$ maps D_R into itself. Now, Perov's fixed point theorem applies and guarantees the existence of a unique fixed point $(u, v) \in D_R$. Then, $x = e^u, y = e^v$ is the unique solution of the Cauchy problem (1)–(2).

4. Continuous dependence of the solution on parameter λ.

Denoting $S_1(\lambda) = u$ and $S_2(\lambda) = v$, we have

$$|S_1(\lambda)(t) - S_1(\mu)(t)| \leq$$

$$\leq \int_0^t \int_0^\tau \left| f(e^{S_1(\lambda)(s)}, e^{S_2(\lambda)(s)}, \lambda) - f(e^{S_1(\mu)(s)}, e^{S_2(\mu)(s)}, \mu) \right| ds\, d\tau$$

$$\leq \int_0^t \int_0^\tau \left(a_{11} |e^{S_1(\lambda)(s)} - e^{S_1(\mu)(s)}| + a_{12} |e^{S_2(\lambda)(s)} - e^{S_2(\mu)(s)}| + a_{13} |\lambda - \mu| \right) ds\, d\tau.$$

Using Lagrange's mean value theorem, we have

$$\left| e^{S_1(\lambda)(s)} - e^{S_1(\mu)(s)} \right| \leq \rho \left| S_1(\lambda)(s) - S_1(\mu)(s) \right|$$

and

$$\left| e^{S_2(\lambda)(s)} - e^{S_2(\mu)(s)} \right| \leq \rho \left| S_2(\lambda)(s) - S_2(\mu)(s) \right|.$$

Then

$$|S_1(\lambda)(t) - S_1(\mu)(t)| \le \int_0^t \int_0^\tau \big(a_{11}\rho\,|S_1(\lambda)(s) - S_1(\mu)(s)| \\ + a_{12}\rho\,|S_2(\lambda)(s) - S_2(\mu)(s)| + a_{13}|\lambda-\mu|\big)\,ds\,d\tau.$$

Now we introduce the Bielecki norm and obtain

$$\begin{aligned}|S_1(\lambda)(t) - S_1(\mu)(t)| &\le \int_0^t \int_0^\tau \big(a_{11}\rho|S_1(\lambda)(s) - S_1(\mu)(s)|e^{-\theta s}e^{\theta s} \\ &\quad + a_{12}\rho|S_2(\lambda)(s) - S_2(\mu)(s)|e^{-\theta s}e^{\theta s} + a_{13}|\lambda-\mu|\big)\,ds\,d\tau \\ &\le a_{11}\rho\,\|S_1(\lambda) - S_1(\mu)\|_\theta \cdot \tfrac{e^{\theta t}}{\theta^2} + a_{12}\rho\,\|S_2(\lambda) - S_2(\mu)\|_\theta \cdot \tfrac{e^{\theta t}}{\theta^2} \\ &\quad + a_{13}|\lambda-\mu| \cdot \tfrac{T^2}{2}.\end{aligned}$$

Multiplying by $e^{-\theta t}$ and taking the maximum, we obtain

$$\|S_1(\lambda) - S_1(\mu)\|_\theta \le \tfrac{a_{11}\rho}{\theta^2}\,\|S_1(\lambda) - S_1(\mu)\|_\theta + \tfrac{a_{12}\rho}{\theta^2}\,\|S_2(\lambda) - S_2(\mu)\|_\theta \\ + a_{13}|\lambda-\mu| \cdot \tfrac{T^2}{2}.$$

Analogously, we have

$$\|S_2(\lambda) - S_2(\mu)\|_\theta \le \tfrac{a_{21}\rho}{\theta^2}\,\|S_1(\lambda) - S_1(\mu)\|_\theta + \tfrac{a_{22}\rho}{\theta^2}\,\|S_2(\lambda) - S_2(\mu)\|_\theta \\ + a_{23}|\lambda-\mu| \cdot \tfrac{T^2}{2}.$$

Writing the above two inequalities in vector form yields

$$\begin{bmatrix} \|S_1(\lambda) - S_1(\mu)\|_\theta \\ \|S_2(\lambda) - S_2(\mu)\|_\theta \end{bmatrix} \le M_\theta \begin{bmatrix} \|S_1(\lambda) - S_1(\mu)\|_\theta \\ \|S_2(\lambda) - S_2(\mu)\|_\theta \end{bmatrix} + \begin{bmatrix} a_{13}|\lambda-\mu|\frac{T^2}{2} \\ a_{23}|\lambda-\mu|\frac{T^2}{2} \end{bmatrix},$$

where

$$M_\theta = \frac{\rho}{\theta^2}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

If θ is chosen large enough so that the entries of the matrix M_θ become sufficiently small, then M_θ is convergent to zero. As a result, the matrix $(I - M_\theta)^{-1}$ exists and belongs to $\mathcal{M}_{2\times 2}(\mathbb{R}_+)$. Therefore, we can multiply both sides of the inequality by this inverse matrix without changing the direction of the inequality. It follows that

$$(I - M_\theta)\begin{bmatrix} \|S_1(\lambda) - S_1(\mu)\|_\theta \\ \|S_2(\lambda) - S_2(\mu)\|_\theta \end{bmatrix} \le \begin{bmatrix} a_{13}|\lambda-\mu|\frac{T^2}{2} \\ a_{23}|\lambda-\mu|\frac{T^2}{2} \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} \|S_1(\lambda) - S_1(\mu)\|_\theta \\ \|S_2(\lambda) - S_2(\mu)\|_\theta \end{bmatrix} \le (I - M_\theta)^{-1}\begin{bmatrix} a_{13}|\lambda-\mu|\frac{T^2}{2} \\ a_{23}|\lambda-\mu|\frac{T^2}{2} \end{bmatrix}.$$

It follows that $(S_1(\lambda), S_2(\lambda))$ depends continuously on the parameter λ. Indeed, if $\mu \longrightarrow \lambda$, then $(S_1(\mu), S_2(\mu)) \to (S_1(\lambda), S_2(\lambda))$. □

The following iterative algorithm is designed to approximate the value of λ corresponding to a solution of the control problem as closely as possible. We are now prepared to present the iterative procedure for solving the control problem.

2.1. The algorithm. Let $(\underline{x}, \underline{y}, \underline{\lambda})$ and $(\overline{x}, \overline{y}, \overline{\lambda})$ be lower and upper solutions of the control problem with $\underline{\lambda} < \overline{\lambda}$.

Step 1. Initialize $\underline{\lambda}_0 := \underline{\lambda}$, $\overline{\lambda}_0 := \overline{\lambda}$, $\underline{x}_0 := \underline{x}$, $\underline{y}_0 := \underline{y}$, $\overline{x}_0 := \overline{x}$, $\overline{y}_0 := \overline{y}$.

Step 2. At any iteration $k \geq 1$, define

$$\lambda_k := \frac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2},$$

and solve problem (1)–(2) for $\lambda = \lambda_k$. One obtains the solution

$$(x_k, y_k) = (e^{S_1(\lambda_k)}, e^{S_2(\lambda_k)}).$$

If $\Psi(x_k(T), y_k(T)) < 0$, then we put

$$\underline{\lambda}_k := \lambda_k,\ \overline{\lambda}_k := \overline{\lambda}_{k-1},$$
$$\underline{x}_k := x_k,\ \overline{x}_k := \overline{x}_{k-1},$$
$$\underline{y}_k := y_k,\ \overline{y}_k := \overline{y}_{k-1},$$

otherwise, for $\Psi(x_k(T), y_k(T)) > 0$, we take

$$\underline{\lambda}_k := \underline{\lambda}_{k-1},\ \overline{\lambda}_k := \lambda_k,$$
$$\underline{x}_k := \underline{x}_{k-1},\ \overline{x}_k := x_k,$$
$$\underline{y}_k := \underline{y}_{k-1},\ \overline{y}_k := y_k$$

and we repeat Step 2 for $k = k + 1$. Obviously, if $\Psi(x_k(T), y_k(T)) = 0$, then we have the solution and we are finished.

The algorithm stops when

$$|\Psi(x_k, y_k)| < \delta,$$

for a given error $\delta > 0$.

Using Theorem 4 we can prove the convergence of the above algorithm.

THEOREM 5. *Under the assumptions of Theorem 4, the algorithm is convergent to a solution of the control problem.*

Proof. Assume that the algorithm does not stop in a finite number of steps. Then it gives a bounded increasing sequence $(\underline{\lambda}_k)$, a bounded decreasing sequence $(\overline{\lambda}_k)$, and the sequences of solutions $(\underline{x}_k, \underline{y}_k)$, $(\overline{x}_k, \overline{y}_k)$, where

$$\underline{x}_k = e^{S_1(\underline{\lambda}_k)},\ \overline{x}_k = e^{S_1(\overline{\lambda}_k)},$$
$$\underline{y}_k = e^{S_2(\underline{\lambda}_k)},\ \overline{y}_k = e^{S_2(\overline{\lambda}_k)},$$

with the following properties:

$$\Psi\big(\underline{x}_k(T), \underline{y}_k(T)\big) < 0,\ \Psi\big(\overline{x}_k(T), \overline{y}_k(T)\big) > 0, \tag{8}$$

and

$$(9) \qquad |\underline{\lambda}_k - \overline{\lambda}_k|_{\mathbb{R}^m} = \frac{1}{2^k}|\underline{\lambda} - \overline{\lambda}|_{\mathbb{R}^m}.$$

The two sequences $(\underline{\lambda}_k)$, $(\overline{\lambda}_k)$ being monotone and bounded are convergent. Moreover, from (9) they have the same limit λ^*. Using the continuity of Ψ and of S_1, S_2 with respect to λ, and (8), we deduce that

$$(10) \qquad \Psi(x^*(T), y^*(T)) = 0,$$

where $x^* := e^{S_1(\lambda^*)}$ and $y^* := e^{S_2(\lambda^*)}$. Then (10) shows that (x^*, y^*, λ^*) is a solution the control problem. □

We next assume that the Cauchy problem can be approximately solved with a desired error ε. In this situation, the algorithm changes as follows.

2.2. The approximate algorithm. Let $\varepsilon > 0$ be an admissible error and $(\underline{\tilde{x}}, \underline{\tilde{y}}, \underline{\tilde{\lambda}})$, $(\overline{\tilde{\tilde{x}}}, \overline{\tilde{\tilde{y}}}, \overline{\tilde{\tilde{\lambda}}})$ be approximate lower and upper solutions of the Cauchy problem with error ε.

Step 1. Initialize $\underline{\lambda}_0 := \underline{\tilde{\lambda}}$, $\overline{\lambda}_0 := \overline{\tilde{\tilde{\lambda}}}$, $\underline{\tilde{x}}_0 := \underline{\tilde{x}}$, $\underline{\tilde{y}}_0 := \underline{\tilde{y}}$, $\overline{\tilde{\tilde{x}}}_0 := \overline{\tilde{\tilde{x}}}$, $\overline{\tilde{\tilde{y}}}_0 := \overline{\tilde{\tilde{y}}}$.

Step 2. At any iteration $k \geq 1$, define

$$\lambda_k := \frac{\underline{\lambda}_{k-1} + \overline{\lambda}_{k-1}}{2},$$

solve approximatively the Cauchy problem and find the approximate solution $(\tilde{x}_k, \tilde{y}_k)$. If $\Psi(\tilde{x}_k(T), \tilde{y}_k(T)) < 0$, then put

$$\underline{\lambda}_k := \lambda_k,\ \overline{\lambda}_k := \overline{\lambda}_{k-1},\ \underline{\tilde{x}}_k := \tilde{x}_k,\ \underline{\tilde{y}}_k := \tilde{y}_k,\ \overline{\tilde{\tilde{x}}}_k := \overline{\tilde{\tilde{x}}}_{k-1},\ \overline{\tilde{\tilde{y}}}_k := \overline{\tilde{\tilde{y}}}_{k-1},$$

otherwise, for $\Psi(x_k(T), y_k(T)) > 0$ take

$$\underline{\lambda}_k := \underline{\lambda}_{k-1},\ \overline{\lambda}_k := \lambda_k,\ \underline{\tilde{x}}_k := \underline{\tilde{x}}_{k-1},\ \underline{\tilde{y}}_k := \underline{\tilde{y}}_{k-1},\ \overline{\tilde{\tilde{x}}}_k := \tilde{x}_k,\ \overline{\tilde{\tilde{y}}}_k := \tilde{y}_k,$$

and we repeat Step 2 for $k = k + 1$.

The algorithm stops if

$$|\Psi(\tilde{x}_k, \tilde{y}_k)| < \delta,$$

for a given error $\delta > 0$.

Theorem 6. *Under the assumptions of Theorem 4 and if in addition Ψ satisfies*

$$|\Psi(t, s) - \Psi(\overline{t}, \overline{s})| \leq L(|t - \overline{t}| + |s - \overline{s}|),$$

for all $t, \overline{t}, s, \overline{s} \in \mathbb{R}$, then the approximate algorithm gives us a triple (x^*, y^*, λ^*), where $\lambda^* = \lim_{k\to\infty} \underline{\lambda}_k = \lim_{k\to\infty} \overline{\lambda}_k$, the pair (x^*, y^*) is the exact solution of Cauchy problems for $\lambda = \lambda^*$, and

$$\Psi(x^*(T), y^*(T)) \in [-2\varepsilon L, 2\varepsilon L].$$

Proof. Denote

$$\underline{x}_k = e^{S_1(\underline{\lambda}_k)},\ \underline{y}_k = e^{S_2(\underline{\lambda}_k)},$$
$$\overline{x}_k = e^{S_1(\overline{\lambda}_k)},\ \overline{y}_k = e^{S_2(\overline{\lambda}_k)},$$

the exact solution pairs of the Cauchy problem corresponding to the numbers $\underline{\lambda}_k, \overline{\lambda}_k$ generated by the approximate algorithm. Clearly, $\|\underline{\tilde{x}}_k - \underline{x}_k\| \le \varepsilon$ and $\|\underline{\tilde{y}}_k - \underline{y}_k\| \le \varepsilon$. Also, for any $\varepsilon' > 0$, there is $k_{\varepsilon'}$ such that

$$\|\underline{x}_k - x^*\| \le \varepsilon', \quad \|\underline{y}_k - y^*\| \le \varepsilon' \text{ for all } k \ge k_{\varepsilon'}.$$

Hence

$$\|\underline{\tilde{x}}_k - x^*\| \le \|\underline{\tilde{x}}_k - \underline{x}_k\| + \|\underline{x}_k - x^*\| \le \varepsilon + \varepsilon', \quad \text{for } k \ge k_{\varepsilon'}$$

and similarly,

$$\left\|\underline{\tilde{y}}_k - y^*\right\| \le \varepsilon + \varepsilon'.$$

Then

$$\Big|\Psi\big(x^*(T), y^*(T)\big) - \Psi\big(\underline{\tilde{x}}_k(T), \underline{\tilde{y}}_k(T)\big)\Big| \le 2L(\varepsilon + \varepsilon'),$$

from which it follows that

$$\begin{aligned}\Psi\big(x^*(T), y^*(T)\big) &\le \Psi\big(\underline{\tilde{x}}_k(T), \underline{\tilde{y}}_k(T)\big) + 2L(\varepsilon + \varepsilon') \\ &< 2L(\varepsilon + \varepsilon').\end{aligned}$$

Analogously, we find

$$\Big|\Psi\big(x^*(T), y^*(T)\big) - \Psi\big(\overline{\tilde{x}}_k(T), \overline{\tilde{y}}_k(T)\big)\Big| \le 2L(\varepsilon + \varepsilon'),$$

whence

$$\begin{aligned}\Psi\big(x^*(T), y^*(T)\big) &\ge \Psi\big(\overline{\tilde{x}}_k(T), \overline{\tilde{y}}_k(T)\big) - 2L(\varepsilon + \varepsilon') \\ &> -2L(\varepsilon + \varepsilon').\end{aligned}$$

In conclusion

$$\Psi\big(x^*(T), y^*(T)\big) \in \big(-2L(\varepsilon + \varepsilon'), 2L(\varepsilon + \varepsilon')\big).$$

Letting $\varepsilon' \to 0$ gives the final conclusion.

$$\Psi\big(x^*(T), y^*(T)\big) \in [-2L\varepsilon, 2L\varepsilon]\,. \tag{11}$$

□

Based on the previous results, we can make the following remark.

Remark 7. (a) *Estimate* (11) *shows that the controllability condition is satisfied with an error of at most* $2\varepsilon L$.

(b) *If* $(\tilde{x}^*, \tilde{y}^*)$ *is an approximate solution corresponding to* $\lambda = \lambda^*$, *with some error* ε, *then*

$$\Psi(\tilde{x}^*(T), \tilde{y}^*(T)) \in [-4\varepsilon L, 4\varepsilon L]. \tag{12}$$

Indeed, we have

$$\begin{aligned}|\Psi(x^*(T), y^*(T)) - \Psi(\tilde{x}^*(T), \tilde{y}^*(T))| &\leq L(|x^*(T) - \tilde{x}^*(T)| + |y^*(T) - \tilde{y}^*(T)|) \\ &\leq 2\varepsilon L.\end{aligned}$$

It follows that

$$\begin{aligned}-4\varepsilon L \leq \Psi(x^*(T), y^*(T)) - 2\varepsilon L &\leq \Psi(\tilde{x}^*(T), \tilde{y}^*(T)) \\ &\leq \Psi(x^*(T), y^*(T)) + 2\varepsilon L \leq 4\varepsilon L,\end{aligned}$$

which proves (12).

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Received by the editors: December 03, 2024; accepted: February 05, 2025; published online: June 30, 2025.