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APPROXIMATION OF CONVEX FUNCTIONS BY CONVEX SPLINES AND CONVEXITY PRESERVING CONTINUOUS LINEAR OPERATORS

by

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1. Introduction and results

Let f be a real-valued function defined on $[a, b]$. By $[f(x_0), \ldots, f(x_l)]$ we mean the l-th divided difference of f:

$$[f(x_0), \ldots, f(x_l)] = \sum_{k=0}^{l} f(x_k)/\omega_l'(x_k) \tag{1.1}$$

where $a \leq x_0 < \ldots < x_l \leq b$ and $\omega_l(x) = (x - x_0) \ldots (x - x_l)$, $l = 0, 1, 2, \ldots$.

The function f is a convex function of order l if $[f(x_0), \ldots, f(x_l)] \geqslant 0$ for every choice of $a \leq x_0 < \ldots < x_l \leq b$. The set of all continuous convex functions of order l on $[a, b]$ will be denoted by $K_l[a, b]$. When convexity problems are discussed, it is customary to assume that $l \geqslant 2$. However, it will be convenient here to allow also the values $l = 1$ and $l = 0$. Consequently, $K_1[a, b]$ will be the set of all continuous, non-decreasing functions on $[a, b]$ and $K_0[a, b]$ will be the set of all continuous, non-negative functions on $[a, b]$.

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Our main result is a generalization of the following geometrically obvious result:

A continuous, non-decreasing function f on $[a, b]$ can be approximated uniformly on $[a, b]$ by functions of the form

$$\psi_n(x) = f(a) + \frac{f(b) - f(a)}{n} \sum_{k=1}^{n-1} (x - c_k)_+^0$$

where $n \geq 2$ and $c_k \in (a, b)$, $k = 1, \ldots, n - 1$, and

$$(x - c)_+^0 = \begin{cases} 0 \text{ if } x < c \\ 1 \text{ if } x \geq c. \end{cases}$$

More precisely, the points c_k, $k = 1, \ldots, n - 1$, can be chosen so that $\psi_n(x) \leq f(x) \leq \psi_n(x) + (f(b) - f(a))/n$ for every $x \in [a, b]$.

THEOREM 1. *Every* $f \in K_l[a, b]$, $l \geq 2$, *can be approximated uniformly on* $[a, b]$ *by spline functions of the form*

$$\psi_{m,n}(f, x) = \sum_{k=0}^{l-1} \binom{l}{k} \Delta_{\frac{b-a}{m}}^k f(a)(x - a)^k/(b - a)^k + \tag{1.2}$$

$$+ \frac{C(f, m)}{n} \sum_{k=1}^{n-1} (x - c_k)_+^{l-1}/(b - a)^{l-1}$$

where $C(f, m) \geq 0$, $m \geq l$, $n \geq 2$, $c_k \in (a, b)$, $k = 1, \ldots, n - 1$ *and*

$$(x - c)_+^{l-1} = \begin{cases} 0 \text{ if } x < c \\ (x - c)^{l-1} \text{ if } x \geq c. \end{cases}$$

In view of the preceding remark, Theorem 1 is true also if $l = 1$. The coefficient $C(f, m)$ is defined by

$$C(f, m) = \binom{m}{l-1} \sum_{k=0}^{m-l} \Delta_{\frac{b-a}{m}}^l f\left(a + k\frac{b-a}{m}\right)$$

and

$$\Delta_h^k f(x) = \sum_{r=0}^{k} (-1)^{k-r} \binom{k}{r} f(x + rh).$$

As an application of this theorem we shall mention two results which characterize convexity preserving continuous linear operators defined on the space $C[a, b]$ of continuous functions on $[a, b]$, with values in $F[a, b]$, the space of bounded real-valued functions on $[a, b]$ endowed with the supremum norm.

The first of these results gives necessary and sufficient conditions for a continuous linear operator to transform every continuous convex function of order $l \geq 2$ into a convex function of order $r \geq 0$.

THEOREM 2. *Let* $\Psi : C[a, b] \to F[a, b]$ *be a continuous linear operator. In order that for every* $f \in K_l[a, b]$, $l \geq 2$, *we have* $\Psi(f, \cdot) \in K_r[a, b]$, $r \geq 0$, *it is necessary and sufficient that*

(i) $[\Psi(P, x_0), \ldots, \Psi(P, x_r)] = 0$ *for every polynomial* P *of degree* $\leq l - 1$ *and every set of* $r + 1$ *points* $x_0 < \ldots < x_r$ *in* $[a, b]$.

(ii) $\Psi((t - c)_+^{l-1}, \cdot) \in K_r[a, b]$ *for every* $c \in (a, b)$.

Clearly, condition (i) means that for every polynomial P of degree $\leq l - 1$, $\Psi(P, \cdot)$ should be identically zero if $r = 0$, or a polynomial of degree $\leq r - 1$ if $r \geq 1$.

The second result gives necessary and sufficient conditions for a continuous linear operator to transform a convex function which is in every class $K_i[a, b]$, $i = 0, 1, \ldots, l$ $(l \geq 2)$ into an element of an arbitrary closed, convex cone $M[a, b] \subseteq C[a, b]$.

THEOREM 3. *Let* $\Psi : C[a, b] \to F[a, b]$ *be a continuous linear operator and let* $M[a, b]$ *be a cloeed convex cone in* $C[a, b]$. *In order that for every* $f \in \bigcap_{i=0}^{l} K_i[a, b]$, $l \geq 2$, *we have* $\Psi(f, \cdot) \in M[a, b]$, *it is necessary and sufficient that*

(i) $\Psi((t - a)^r, \cdot) \in M[a, b]$ *for every* $0 \leq r \leq l - 1$

(ii) $\Psi((t - c)_+^{l-1}, \cdot) \in M[a, b]$ *for every* $c \in (a, b)$.

Theorems 2 and 3 are true also if $l = 1$ if one assumes that $\Psi((t - c)_+^0, \cdot)$ has a meaning since $(t - c)_+^0 = \chi_{[c, \infty)}(t)$ is not continuous on $[a, b]$ if $c \in (a, b)$.

Problems such as these have been studied by several authors. First results of this type were obtained by T. POPOVICIU [1]. He has studied monotonicity preserving operators of the form

$$\Phi(x, f) = \sum_{i=1}^{n} f(\xi_i)\, \varphi_i(x)$$

where $a \leq \xi_1 < \ldots < \xi_n \leq b$ and φ_i, $i = 1, \ldots, n$ are differentiable functions satisfying the condition

$$\sum_{i=1}^{n} \varphi_i'(x) = 1, \; x \in [a, b].$$

Under these hypotheses, in view of Theorem 2, a necessary and sufficient condition for Φ to transform a non-decreasing function into a non-decreasing function is that for every $c \in (a, b)$ the function

$$\Phi((t-c)_+^0, x) = \sum_{i=1}^{n} (\xi_i - c)_+^0 \varphi_i(x)$$

be a non-decreasing function on $[a, b]$. Since the functions φ_i, $i = 1, \ldots, n$ are differentiable functions, this will be true if

$$(\Phi((t-c)_+^0, x)' = \sum_{i=1}^{n} (\xi_i - c)_+^0 \varphi_i'(x) \geqslant 0,$$

which is equivalent to Popoviciu's condition

$$\sum_{i=1}^{j} \varphi_i'(x) \leq 1 \text{ for } j = 1, \ldots, n$$

Similar results were obtained by J. A. ROULIER [2] for operators of the form

$$\Phi(f, x) = \int_a^b f(t)\, K(x, t)\, dt$$

where K is a continuous function on $[a, b] \times [a, b]$. He has proved that $\Phi(f, \cdot)$ is a non-negative increasing function on $[a, b]$ for every continuous, non-negative and increasing function f if and only if

$$\Phi((t-c)_+^0, x) = \int_c^b K(x, t)\, dt$$

is non-negative and increasing for every $c \in [a, b]$. This result is clearly a simple consequence of Theorem 3.

Using a more general definition of convexity and an essentially different approach, Z. ZIEGLER [3] and S. KARLIN and W. J. STUDDEN [4, Ch. 11] have developed a theory of generalized convex functions which contains results similar to Theorems 1—3. Our proof of Theorem 1 is based on the fact, observed first by T. POPOVICIU [5], that Bernstein polynomial $B_m(f, \cdot)$ of a convex function f of order l is also a convex function of order l, and on the well-known fact that $B_m f, x) \to f(x)$ uniformly as $m \to \infty$. These two facts. for which, apparently, there is no obvious analogy in the theory of generalized convex functions, make our proof of Theorem 1 very simple. Theorems 2 and 3 follow then easily from the Theorem 1.

2. Proofs.

We shall, for simplicity, prove Theorem 1 only for the special interval [0, 1]; the general case then follows by simple substitutions. The proof of Theorem 1 is based on the following lemma.

Lemma 1. *Let g be l-times differentiable on $[0, 1]$, $l \geqslant 2$, and let $g^{(l)}(x) \geqslant 0$ for $x \in [0, 1]$. Let*

$$T_{l-1}(g, x) = \sum_{k=0}^{l-1} \frac{x^k}{k!} g^{(k)}(0)$$

be the Taylor polynomial of g of degree $\leq l-1$. Then for every $n \geqslant 2$ there exist points $c_1 < \ldots < c_{n-1}$ in $(0, 1)$ such that

$$g(x) = T_{l-1}(g, x) + \frac{g^{(l-1)}(1) - g^{(l-1)}(0)}{n(l-1)!} \sum_{k=1}^{n-1} (x - c_k)_+^{l-1} + R_n(g, x)$$

where

$$|R(g, x)| \leq \frac{1}{n(l-1)!} \int_0^1 g^{(l)}(t)\, dt.$$

Proof of Lemma 1. We have first

$$g(x) = T_{l-1}(g, x) + \frac{1}{(l-2)!} \int_0^1 (x-t)_+^{l-2} (g^{(l-1)}(t) - g^{(l-1)}(0))\, dt. \tag{2.1}$$

Since $g^{(l)}(x) \geqslant 0$ on $[0, 1]$, the function $g^{(l-1)}(x) - g^{(l-1)}(0)$ is non-negative, non-decreasing and continuous on $[0, 1]$. Hence we can find $c_1 < \ldots < c_{n-1}$ in $(0, 1)$ such that

$$\left| g^{(l-1)}(t) - g^{(l-1)}(0) - \frac{g^{(l-1)}(1) - g^{(l-1)}(0)}{n} \sum_{k=1}^{n-1} (t - c_k)_+^0 \right| \leqslant \frac{g^{(l-1)}(1) - g^{(l-1)}(0)}{n}$$

and it follows that, for every $t \in [0, 1]$,

$$g^{(l-1)}(t) - g^{(l-1)}(0) - \frac{g^{(l-1)}(1) - g^{(l-1)}(0)}{n} \sum_{k=1}^{n-1} (t - c_k)_+^0 + \varepsilon_n(g, t) \tag{2.2}$$

where

$$|\varepsilon_n(g, t)| \leq \frac{g^{(l-1)}(1) - g^{(l-1)}(0)}{n} = \frac{1}{n} \int_0^1 g^{(l)}(u)\, du. \tag{2.3}$$

The proof of Lemma 1 is finally completed by substituting (2.2) into (2.1) and using inequality (2.3).

Proof of Theorem 1. Let $B_m(f, \cdot)$ be the Bernstein polynomial of f of degree $m \geqslant l$:

$$B_m(f, x) = \sum_{k=0}^{m} f\left(\frac{k}{m}\right)\binom{m}{k} x^k (1-x)^{m-k}.$$

Since

(2.4)
$$B_m^{(l)}(f, x) = m(m-1)\ldots(m-l+1)\sum_{k=0}^{m-l} \Delta^l_{1/m} f\left(\frac{k}{m}\right)\binom{m-l}{k} x^k (1-x)^{m-l-k}$$

and since $f \in K_l[0, 1]$, we see that $B_m^{(l)}(f, x) \geqslant 0$ for $x \in [0, 1]$. By Lemma 1 we can find points $c_1 < \ldots < c_{n-1}$ in $(0, 1)$ such that, for every $x \in [0. 1]$,

$$(2.5) \quad B_m(f, x) = T_{l-1}(B_m(f), x) + \frac{C(f, m)}{n}\sum_{k=1}^{n-1}(x - c_k)_+^{l-1} + R_n(B_m(f), x)$$

where

$$(2.6) \qquad C(f, m) = \frac{B_m^{(l-1)}(f, 1) - B_m^{(l-1)}(f, 0)}{(l-1)!} = \frac{1}{(l-1)}\int_0^1 B_m^{(l)}(f, t)\, dt \geqslant 0$$

and, by Lemma 1,

$$(2.7) \qquad |R_n(B_m(f), x)| \leq \frac{1}{n(l-1)!}\int_0^1 B_m^{(l)}(f, t)\, dt = \frac{C(f, m)}{n}.$$

Let

$$(2.8) \qquad \psi_{m,n}(f, x) = T_{l-1}(B_m(f), x) + \frac{C(f, m)}{n}\sum_{k=1}^{n-1}(x - c_k)_+^{l-1}.$$

In view of (2.5) and (2.7) we have

$$|B_m(f, x) - \psi_{m,n}(f, x)| \leq \frac{C(f, m)}{n}$$

and it follows that, for every $x \in [0, 1]$,

$$|f(x) - \psi_{m,n}(f, x)| \leq |f(x) - B_m(f, x)| + |B_m(f, x) - \psi_{m,n}(f, x)|$$
$$\leq 2\omega_f\left(\frac{1}{\sqrt{m}}\right) + \frac{C(f, m)}{n}.$$

Hence, by choosing first m large enough so that $\omega_f(1/\sqrt{m})$ is small, and then choosing n so large that $C(f, m)/n$ is small, we see that any $f \in K_l[0, 1]$, $l \geqslant 2$, can be approximated uniformly on $[0, 1]$ by spline functions $\psi_{m,n}(f, \cdot)$. In order to complete the proof of Theorem 1 we have only to observe that

$$T_{l-1}(B_m(f), x) = \sum_{k=0}^{l-1}\binom{l}{k}(\Delta^k_{1/m} f(0))\, x^k$$

and that, by (2.6) and (2.4),

$$C(f, m) = \frac{m(m-1)\ldots(m-l+1)}{(l-1)!}\sum_{k=0}^{m-l}\Delta^l_{1/m} f(k/m)\binom{m-l}{k}\int_0^1 t^k(1-t)^{m-l-k}\, dt$$
$$= \binom{m}{l-1}\sum_{k=0}^{m-l}\Delta^l_{1/m} f(k/m).$$

It follows then from (2.8) that

$$\psi_{m,n}(f, x) = \sum_{k=0}^{l-1}\binom{l}{k}(\Delta^k_{1/m} f(0))\, x^k + \frac{C(f, m)}{n}\sum_{k=1}^{n-1}(x - c_k)_+^{l-1}$$

where $C(f, m) \geqslant 0$ by (2.6).

Proof of Theorem 2. (Necessity) Suppose that Ψ transforms every convex function of order $l \geqslant 2$ into a convex function of order $r \geqslant 0$. Let P be a polynomial of degree $\leq l - 1$. Since P and $-P$ are both convex functions of order l, the functions $\Psi(\pm P, \cdot)$ are convex functions of order $r \geqslant 0$. Hence

$$[\Psi(\pm P, x_0), \ldots, \Psi(\pm P, x_r] \geqslant 0$$

and (i) follows. The necessity of (ii) is obvious since $(x - c)_+^{l-1} \in K_l[a, b]$, $l \geqslant 2$.

(Sufficiency) Let $f \in K_l[a, b]$ and let $\Psi_{m,n}(f, \cdot)$ be a spline function of the form (1.2). The proof of sufficiency of conditions (i) and (ii) of Theorem 2 consists in showing that these conditions and $C(f, m) \geqslant 0$ imply

$$[\Psi_{m,n}(\psi, x_0), \ldots, \Psi_{m,n}(\psi, x_r)] \geqslant 0.$$

Using the fact that Ψ is a continuous linear operator and that the functions $\psi_{m,n}(f, \cdot)$ approximate uniformly f on $[a, b]$, we obtain

$$[\Psi(f, x_0), \ldots, \Psi(f, x_r)] \geqslant 0$$

and so $\Psi(f, \cdot) \in K_r[a, b]$.

The proof of Theorem 3 is similar. Necessity of conditions (i) and (ii) is obvious. To show that these conditions are sufficient, observe that the hypothesis $f \in \bigcap_{i=0}^{l} K_i[a, b]$ implies that all the coefficients of the function $\psi_{m,n}(f, \cdot)$ in (1.2) are non-negative. Conditions (i) and (ii) of Theorem 3 imply then that $\Psi(\psi_{m,n}, \cdot) \in M[a, b]$ and the rest of the proof is completed as before by observing that Ψ is a continuous linear operator and that the functions $\psi_{m,n}(f, \cdot)$ approximate uniformly f on $[a, b]$. Since the cone $M[a, b]$ is closed, by hypothesis, this implies that $\Psi(f, \cdot) \in M[a, b]$ and Theorem 3 is proved.

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