MATHEMATICA — REVUE D'ANLAYSE NUMÉRIQUE
ET DE THÉORIE DE L'APPROXIMATION

L'ANALYSE NUMÉRIQUE ET LA THÉORIE DE L'APPROXIMATION
Tome 8, N⁰ 1, 1979, pp. 83—87

APPROXIMATION OF COMMON FIXED POINT IN A GENERALIZED METRIC SPACE

by

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0. Introduction. Let X be a nonempty set. By definition, a generalized metric on X is a mapping $d: X \times X \to \mathbf{R}^n$ such that

(i) $d(x, y) \geqslant 0$, $\forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$;

(ii) $d(x, y) = d(y, x)$, $\forall x, y \in X$;

(iii) $d(x, y) \leqslant d(x, z) + d(z, y)$, $\forall x, y, z \in X$.

By a generalized metric space we mean an entity (X, d) consisting of a nonempty set X and a generalized metric d on X.

Exemple 1. $X = \mathbf{R}^n$, $d(x, y) = (|x_1 - y_1|, \ldots, |x_n - y_n|)$

Exemple 2. $X = C([a, b]), \mathbf{R}^n)$, $d(f, g) = (\|f_1 - g_1\|_{C[a,b]}, \ldots, \|f_n - g_n\|_{C[a,b]}$.

Exemple 3. $X = C([a, b], \mathbf{R}^2)$, $d(f, g) = (\|f_1 - g_1\|_{C[a,b]}, \|f_2 - g_2\|_{L^2(a,b)})$.

Several fixed point theorems are known (see [1]—[3]), [5]) for mappings in generalized metric space.

In this note we prove the following common fixed point theorem

THEOREM 1. *Let* (X, d) *be a complet generalized metric space* ($d(x,y) \in \mathbf{R}^k$) *and* f, $g: X \to X$ *two mappings for which there exists* $A \in M_{kk}(\mathbf{R}_+)$, $[(I - A)^{-1}A]^n \to 0$, *when* $n \to +\infty$, *such that*

$$d(f(x), g(y)) \leqslant A[d(x, f(x)) + d(y, g(y))], \text{ for all } x, y \in X.$$

Then

(a) $F_f = F_g = \{x^*\}$

(b) for each $x_0 \in X$, the sequence $(x_n)_{n \in IN}$ defined by

$$x_{2n} = (gf)^n(x_0), \quad x_{2n+1} = f(x_{2n})$$

converges to x^* and

$$d(x_n, x^*) \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, x_1)$$

(c) for each $x_0 \in X$, $f^n(x_0) \to x^*$, $g^n(x_0) \to x^*$, when $n \to \infty$ and

$$d(f(x_0), x^*) \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, f(x_0)),$$

$$d(g(x_0), x^*) \in (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, g(x_0)).$$

(d) Let $h: X \to X$ be a mapping which approximates the mapping f, and $s: X \to X$ aproximates the mapping g, and

$$d(f(x), h(x)) \leqslant \eta, \ \forall x \leqslant X$$

$$d(g(x), s(x)) \leqslant \eta, \ \forall x \in X$$

If $y_n = h^n(x_0)$, then

$$d(y_n, x^*) \leqslant (I - A)(I - 2A)^{-1}\eta +$$
$$+ 2(n - 1)[(I - A)^{-1}A]^n d(x_0, f(x_0)) +$$
$$+ (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, f(x_0))$$

(e) If $f_n, g_n: X \to X$, are such that $f_n \rightrightarrows f$, $g_n \rightrightarrows g$ and $a_n \in F_{f_n}$, $b_n \in F_{g_n}$, then

$$a_n \to x^*, \ b_n \to x^*, \text{ when } n \to \infty$$

1. *Proof of theorem* 1

(a) + (b). Let x_0 be any element of X. We have

$$d(x_1, x_2) = d(f(x_0), g(x_1)) \leqslant A[d(x_0, x_1) + d(x_1, x_2)]$$

and

$$d(x_1, x_2) \leqslant (I - A)^{-1}A d(x_0, x_1).$$

By induction

$$d(x_n, x_{n-1}) \leqslant [(I - A)^{-1}A]^n d(x_0, x_1).$$

Hence

$$d(x_n, x_{n+p}) \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, x_1),$$

which implies that the sequence $(x_n)_{n \in IN}$ is fundamental. Let x^* be the limit of this sequence. We have

$$x^* \in F_f \cap F_g = F_f = F_g = \{x^*\}$$

and

$$d(x_n, x^*) \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, x_1)$$

(c). Ve have for all $x \in X$:

$$d(f(x), f^2(x) \leqslant d(f(x), g(x^*)) + d(f^2(x), g(x^*)) \leqslant$$
$$\leqslant A[d(x, f(x)) + d(x^*, g(x^*))] + A[d(f(x), f^2(x)) + d(x^*, g(x^*))] =$$
$$= A[d(x, f(x)) + Ad(f(x), f^2(x))].$$

Hence

$$d(f(x), f^2(x)) \leqslant (I - A)^{-1}Ad(x, f(x)), \ \forall x \in X,$$

this implies

$$d(f^n(x_0), f^{n+p}(x_0)) \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, f(x_0))$$

and the sequence $(f^n(x_0))_{n \in IN}$ is fundamental. Let x_f^* be the limit. We have:

$$d(f^n(x_0), x_f^* \leqslant (I - A)(I - 2A)^{-1}[(I - A)^{-1}A]^n d(x_0, f(x_0)).$$

On the other hand from

$$d(f^n(x_0), x^*) \leqslant Ad(f^{n-1}(x_0), f^n(x_0))$$

we have

$$\lim_{n \to} f^n(x_0) = x^* = x_f^*.$$

In a similar way we prove that the sequence $g^n(x_0)$ converges to x^*.

(d). We have:

$$d(y_n, x^* \leqslant d(y_n, f^n(x_0)) + d(f^n(x_0), x^*),$$
$$d(y_n, f^n(x_0)) \leqslant \eta + d(g(y_{n-1}), f^n(x_0)) \leqslant$$
$$\leqslant \eta + Ad(y_{n-1}), g(y_{n-1})) + Ad(f^{n-1}(x_0), f^n(x_0))$$

and

$$d(y_n, g(y_n)) \leqslant (I - A)^{-1}d(y_n, f^n(x_0)) + Ad(f^{n-1}(x_0), f^n(x_0)).$$

From these we have by induction

$$d(y_n, f^n(x_0)) \leqslant (I - A)(I - 2A)^{-1}\eta +$$
$$+ 2(n - 1)[(I - A)^{-1}A]^n d(x_0, f(x_0)).$$

(e). We have

$$d(a_n, x^*) = d(f_n(a_n), f(x^*)) \leqslant d(f_n(a_n), f(y_n)) + d(f(a_n), g(x^*)) \leqslant$$
$$\leqslant d(f_n(a_n), f(a_n)) + A[d(a_n, f(a_n)) + d(x^*, g(x^*))] \leqslant$$
$$\leqslant (I + A)\, d(f_n(a_n), f(a_n))$$

and $d(a_n, x^*) \to 0$, when $n \to \infty$.

In the similar way we prove that

$$b_n \to x^*, \text{ when } n \to \infty.$$

2. The case when $d(x, y) \in \mathbf{R}$. In the case when $d(x, y) \in \mathbf{R}$ from theorem 1 we have:

THEOREM 2. (X, d) be a complete metric space and $f, g : X \to X$ two mapping and $\alpha \in \left[0, \frac{1}{2}\right[$. If

$$d(f(x), g(y)) \leqslant \alpha[d(x, f(x)) + d(y, g(y))], \ \forall x, y \in X,$$

then

(i) (Kannan) : $F_f = F_g = \{x^*\}$

(ii) (Kannan) : For each $x_0 \in X$, the sequence defined by

$$x_{2n} = (gf)^n(x_0), \quad x_{2n+1} = f(x_{2n})$$

converges to x^* and

$$d(x_n, x^*) \leqslant \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, x_1)$$

(iii) (new result) : For each $x_0 \in X$, $f^n(x_0) \to x^*$, $g^n(x_0) \to x^*$ and

$$d(f^n(x_0), x^*) \leqslant \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, f(x_0))$$

$$d(g^n(x_0), x^*) \leqslant \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, g(x_0))$$

(iv) (see [4] and [6]). Let $h, s : X \to X$ be such that

$$d(f(x), h(x)) \leqslant \eta, \quad \forall x \in X$$

$$d(g(x), s(x)) \leqslant \eta, \quad \forall x \in X$$

If $y_n = h^n(x_0)$, $f_n = s^n(x_0)$, then

$$d(y_n, x^*) \leqslant \frac{1-\alpha}{1-2\alpha}\eta + 2(n-1)\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, f(x_0)) +$$
$$+ \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, f(x_0)).$$

and

$$d(z_n, x^*) \leqslant \frac{1-\alpha}{1-2\alpha}\eta + 2(n-1)\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, g(x_0)) +$$
$$+ \frac{1-\alpha}{1-2\alpha}\left[\frac{\alpha}{1-\alpha}\right]^n d(x_0, g(x_0))$$

(v) (see [5]). If $f_n, g_n : X \to X$, are such that

$$f_n \rightrightarrows f, \ g_n \rightrightarrows g \text{ and } a_n \in F_{f_n}, \ b_n \in F_{g_n}, \text{ then}$$

$$a_n \to x^*, \ b_n \to x^*, \text{ when } n \to \infty$$

3. Open problems. Is theorem 1 true for the mappings which satisfy one of the conditions

(1) $(d(x, y) \in \mathbf{R})$. There exist $\alpha, \beta \in \mathbf{R}_+$, $\alpha + 2\beta < 1$, such that

$$d(f(x), g(y)) \leqslant \alpha d(x, y) + \beta[d(x, (fx)) + d(y, g(y))]$$

(2) $(d(x, y) \in \mathbf{R})$. There exist $\alpha, \beta, \gamma \in \mathbf{R}_+$, $\alpha + 2\beta + 2\gamma < 1$, such that

$$d(f(x), g(y)) \leqslant \alpha d(x, y) + \beta[d(x, f(x)) + d(y, g(y))] +$$
$$+ \gamma[d(x, g(y)) + d(y, f(x))]$$

(3) $(d(x, y) \in \mathbf{R}^k)$. There exist $A, B, C \in M_{kk}(\mathbf{R}_+)$, $[(I - B - C)^{-1}(A + B + C)]^n \to 0$, when $n \to \infty$, such that

$$d(f(x), g(y)) \leqslant Ad(x, y) + B[d(x, f(x)) + d(y, g(y))] +$$
$$+ C[d(x, g(y)) + d(y, f(x))].$$

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Received 10.X.1978

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