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SADDLEPOINT NECESSARY CRITERIA FOR NONLINEAR PROGRAMMING IN COMPLEX SPACE

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1. Introduction. Consider the problem

(P) Minimize Re $f(z)$ subject to $z \in M$, $g(z) \in S$,

where M is a nonempty set in C^n, S is a nonempty set in C^m and $f: M \to C$ and $g: M \to C^m$ are functions.

Abrams [1] and Duca [6] gave saddlepoint optimality criteria for a nonlinear programming problem in a complex space. In the proofs of the necessary optimality conditions, it is essential that problem (P) be convex. In his proof, Abrams requested in addition that the functions f and g be analytic.

In this paper, we shall show that the convexity hypothesis can be weakened.

2. Notation and Preliminaries. Let C^n (R^n) denote the n-dimensional complex (real) vector space with Hermitian (Euclidean) norm $\|.\|$, $R^n_+ = \{x \in R^n : x = (x_j)$ with $x_j \geqslant 0$ for all $j \in \{1, \ldots, n\}\}$ the non-negative orthant of R^n, and $C^{m \times n}$ the set of $m \times n$ complex matrices. If A is a matrix or a vector, then A^T, $\bar{A}$, A^H denote its transpose, complex conjugate and conjugate transpose, respectively. For $z = (z_j)$, $w \in C^n$: $\langle z, w \rangle = w^H z$ denotes the inner product of z and w and Re $z = (\text{Re}\, z_j) \in R^n$ denotes the real part of z.

A set S in C^m is said to be a polyhedral cone if it is a finite intersection of closed half-spaces in C^m, each containing 0 in its boundary, i.e.there exists a natural number q and q points $u^1 \ldots, u^q$ in C^m such that

$$S = \cap \{H(u^k) : k \in \{1, \ldots, q\}\},$$

where $H(u^k) = \{v \in C^m : \text{Re} \langle v, u^k \rangle \geqslant 0\}$, $k \in \{1, \ldots, q\}$.

If $S = \cap \{H(u^k) : k \in \{1, \ldots, q\}\}$ is a polyhedral cone in C^m and $v \in C^m$, then we denote

$$S(v) = \cap \{H(u^k) : \text{Re} \langle v, u^k \rangle = 0\}.$$

The polar cone of a nonempty set $S \subseteq C^m$, denoted by S^*, is the set of all u in C^m such that Re $\langle z, u \rangle \geqslant 0$ for each $z \in S$, i.e.

$$S^* = \{u \in C^m : z \in S \Rightarrow \text{Re} \langle z, u \rangle \geqslant 0\}.$$

Let M be an open set in C^n. The function $f : M \to C^m$ is differentiable at $z^0 \in M$ if

$$f(z) - f(z^0) = [\nabla_z f(z^0)]^T(z - z^0) + [\nabla_{\bar z} f(z^0)]^T + (\bar z - \bar z^0)\, o(\|z - z^0\|),$$

for all $z \in M$, where $\nabla_z f(z^0)$ and $\nabla_{\bar z} f(z^0)$ denote, respectively, the $n \times m$ matrices whose k, j th elements are the partial derivatives

$$\frac{\partial f_k(z^0)}{\partial z_j} \text{ and } \frac{\partial f_k(z^0)}{\partial \bar z_j},\ k = 1, \ldots, m;\ j = 1, \ldots, n,$$

and $o(\|z - z^0\|) / \|z - z^0\| \to 0$ as $z \to z^0$.

DEFINITION 1. Let M be a nonempty set in C^n and let S be a nonempty set in C^m. The function $f : M \to C^m$ is said to be:

i) convex at $z \in M$ with respect to S if

$$\left.\begin{array}{l} v \in M,\ v \neq z \\ t \in]0,1[\\ (1-t)z + tv \in M. \end{array}\right\} \Longrightarrow (1-t)f(z) + tf(v) - f[(1-t)z + tv] \in S;$$

ii) concave at $z \in M$ with respect to S if f is convex at $z \in M$ with respect to $-S = \{u \in C^m : -u \in S\}$;

iii) convex (concave) on M with respect to S if M is convex and f is convex (concave) at any $z \in M$ with respect to S.

When referring to the objective function of a programming problem in a complex space, the convexity of the real part is of interest.

DEFINITION 2. Let M be a nonempty set in C^n and let T be a nonempty set in R^m. The function $f : M \to C^m$ is said to have convex real part at $z \in M$(on M) with respect to T if f is convex at z(on M) with respect to $CT = \{u \in C^m : \operatorname{Re} u \in T\}$.

The following result may be found in Duca's [7]:

THEOREM 1. *Let M be a nonempty open set in C^n, let $z \in M$, let S be a closed convex cone in C^m and let $f : M \to C^m$ be a differentiable function at z.*

If f is convex at z with respect to S, then

$$f(v) - f(z) - [\nabla_z f(z)]^T(v - z) - [\nabla_{\bar z} f(z)]^T(\bar v - \bar z) \in S, \text{ for all } z \in M.$$

We shall also need the following result:

THEOREM 2. *Let M be a nonempty open set in C^n, let $z^0 \in M$, let S be a polyhedral cone in C^m with nonempty interior and let $f : M \to C$ and $g : M \to C^m$ be differentiable functions at z^0. If z^0 is a local optimal solution of problem* (P), *then there exist $r \in R$ and $u^0 \in C^m$ such that*

$$r \in R_+,\ u^0 \in (S(g(z^0)))^* \subseteq S^*,\ (r, u^0) \neq (0,\ 0), \tag{1}$$

$$\operatorname{Re}\langle g(z^0), u^0\rangle = 0, \tag{2}$$

$$r\overline{\nabla_z f(z^0)} + r\nabla_{\bar z} f(z^0) - \overline{\nabla_z g(z^0)u^0} - \nabla_{\bar z} g(z^0)\bar u^0 = 0. \tag{3}$$

If, in addition, one of the following conditions holds,

1° *g satisfies the Arrow-Hurwicz-Uzawa complex constraint qualification (CCQ) at z^0 with respect to $X = \{z \in M : g(z) \in M\}$;*

2° *g satisfies the Kuhn-Tucker CCQ at z^0 with respect to X;*

3° *g satisfies the reverse-concave CCQ at z^0 with respect to X;*

4° *g satisfies the weak CCQ at z^0 with respect to X;*

5° *g satisfies Slater's CCQ with respect to X, and g is concave at z^0 with respect to $S(g(z^0))$;*

6° *g satisfies the strict CCQ with respect to X and g is concave at z^0 with respect to $S(g(z^0))$;*

7° *M is convex, g is concave on M with respect to S and g satisfies Karlin's CCQ with respect to X,*

then $r > 0$.

The *proof* is given in ref. [5, Theorems 2 and 4].

3. Results. For $r \in R$, let $L_r : M \times C^m \to C$ denote the function defined by the formula:

$$L_r(z, u) = rf(z) - \langle g(z), u\rangle \quad \text{for all} \quad (z, u) \in M \times C^m.$$

THEOREM 3. *Let M be a nonempty open set in C^n, let $z^0 \in M$ and let S be a polyhedral cone in C^m with nonempty interior. Let $f : M \to C$ be a differentiable function at z^0 having convex real part at z^0 with respect to R_+ and let $g : M \to C^m$ be a differentiable function at z^0 concave at z^0 with respect to S. If z^0 is a local optimal solution of problem* (P), *then there exist $r \in R$ and $u^0 \in C^m$ such that*

$$r \in R_+,\ u^0 \in (S(g(z^0)))^* \subseteq S^*,\ (r, u^0) \neq (0, 0), \tag{4}$$

$$\operatorname{Re}\langle g(z^0), u^0\rangle = 0, \tag{5}$$

$$\operatorname{Re} L_r(z^0, u) \leqslant \operatorname{Re} L_r(z^0, u^0) \leqslant \operatorname{Re} L_r(z, u^0), \tag{6}$$

for all $z \in M$ and $u \in S^*$.

If, in addition, one of the following conditions hold,

1° *g satisfies the Arrow-Hurwicz-Uzawa complex constraint qualification (CCQ) at z^0 with respect to $X = \{z \in M : g(z) \in S\}$;*

2° *g satisfies the Kuhn-Tucker CCQ at z^0 with respect to X;*

3° *g satisfies the reverse-concave CCQ at z^0 with respect to X;*

4° *g satisfies the weak CCQ at z^0 with respect to X;*

5° *g satisfies Slater's CCQ with respect to X;*

6° *g satisfies the strict CCQ with respect to X;*

7° *M is convex, g is concave on M with respect to S and g satisfies Karlin's CCQ with respect to X,*

then $r > 0$.

Proof. In view of theorem 2, there exist $r \in R$ and $u^0 \in C^m$ so that (1) — (3) hold true. Since (4) is (1) and (5) is (2), inequalities (6) need to be demonstated.

Since $g(z^0) \in S$, from (2) we have

$$\operatorname{Re} L_r(z^0, u) = \operatorname{Re}[rf(z^0) - \langle g(z^0), u\rangle] \leqslant \operatorname{Re} rf(z^0) = \\ = \operatorname{Re}[rf(z^0) - \langle g(z^0), u^0\rangle] = \operatorname{Re} L_r(z^0, u^0),$$

for all $u \in S^*$. Hence, the first inequality of (6) holds.

On the other hand, the function f has convex real part at z^0 with respect to R_+ and it is differentiable at z^0; then by theorem 1, we have

$$\operatorname{Re}[f(z) - f(z^0)] \geqslant \operatorname{Re}\{[\nabla_z f(z^0)]^T (z - z^0) + [\nabla_{\bar z} f(z^0)]^T (\bar z - \bar z^0)\}$$

for all $z \in M$.

Since $r \in R_+$, from the latter inequality and the properties of the inner product, we deduce that

$$\text{(7)} \qquad \operatorname{Re} r[f(z) - f(z^0)] \geqslant \operatorname{Re}\langle \overline{r\nabla_z f(z^0) + r\nabla_{\bar z} f(z^0)}, z - z^0\rangle,$$

for all $z \in M$.

The function g is differentiable at z^0 and concave at z^0 with respect to S; then, by theorem 1, we have

$$\text{(8)} \qquad \operatorname{Re}\langle g(z) - g(z^0), u^0\rangle \leqslant \operatorname{Re}\langle \overline{\nabla_z g(z^0) u^0 + \nabla_{\bar z} g(z^0)\bar u^0}, z - z^0\rangle,$$

for all $z \in M$, because $u^0 \in S^*$.

Now, from (7), (8) and (3), we obtain

$$\operatorname{Re} L_r(z, u^0) - \operatorname{Re} L_r(z^0, u^0) = \operatorname{Re}\{r[f(z) - f(z^0)] - \langle g(z) - g(z^0), u^0\rangle\} \geqslant \\ \geqslant \operatorname{Re}\langle \overline{r\nabla_z f(z^0) + r\nabla_{\bar z} f(z^0) - \nabla_z g(z^0) u^0 - \nabla_{\bar z} g(z^0)\bar u^0}, z - z^0\rangle = 0,$$

for all $z \in M$, i.e. the second inequality of (6) holds.

If, in addition, one of the conditions $1^0 - 7^0$ holds, then $r > 0$. This completes the proof.

Example. Consider the problem

$$\text{(9)} \qquad \text{Minimize } (z + \bar z - 2)^3 (z + \bar z - 6)$$

subject to

$$\text{(10)} \qquad -z\bar z + 4z + 4\bar z - 15 \in \{u \in C : -\pi/4 \leqslant \arg u \leqslant \pi/4\}.$$

Let $M = C$, $S = \{u \in C : -\pi/4 \leqslant \arg u \leqslant \pi/4\}$ and let $f(z) = (z + \bar z - 2)^3 (z + \bar z - 6)$ and $g(z) = -z\bar z + 4z + 4\bar z - 15$ for all $z \in M$. Then problem (9) — (10) is of the form (P). Let $z^0 = 3$. Evidently, the functions f and g are differentiable at z^0, the function f has convex real part at z^0 with respect to R_+, the function g is concave at z^0 with respect to S and S is a polyhedral cone in C with nonempty interior. It can be easily verified that z^0 is a local optimal solution of problem (9) — (10). Hence, the hypotheses of theorem 3 are satisfied. Then there exist $r \in R$ and $u^0 \in C$ such that (4) — (6) hold. It can be easily shown that $r = 1$ and $u^0 = 64$.

We remark that for problem (9) — (10), theorem 2 from ref. [1] and theorems 1 and 2 from ref. [6] cannot be applied, because f fails to have convex real part on M with respect to R_+ (the definition is not fulfilled for $z = 1$, $v = 2$ and $t = 1/2$).

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