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FINITELY DEFINED FUNCTIONALS AND DIVIDED DIFFERENCES

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Dedicated to Professor Elena Popoviciu on the occasion of her 80th birthday.

Abstract. We give a necessary and sufficient condition for representing finitely defined functionals in terms of divided differences. As particular cases we obtain formulas of Tiberiu Popoviciu, Newton, etc.

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1. PRELIMINARIES AND NOTATIONS

Let n be a positive integer. We use the following notations and abbreviations:

- $\mathcal{P}_n$, the linear space of polynomials of degree at most n;
- $e_i \in \mathcal{P}_n$, $e_i(t) = t^i$, $i = 0, \ldots, n$;
- $x_0, \ldots, x_n$, distinct points of an interval $[a, b]$;
- $\mathcal{F}$, the linear space of all real functions defined on $\{x_0, \ldots, x_n\}$;
- $\mathcal{A}_n$, a set of linear finitely defined functionals,

$$\mathcal{A}_n = \left\{ A : \mathcal{F} \to \mathbb{R} \;\middle|\; A(f) = \sum_{i=0}^{n} a_i f(x_i), \quad a_0, \ldots, a_n \in \mathbb{R} \right\};$$

- $[t_1, \ldots, t_n; f]$, the divided difference of the function f, with respect to the distinct nodes $t_1, \ldots, t_n$;
- $\{x_{i,0}, \ldots, x_{i,n_i}\}$, $i = 0, \ldots, n$, nonempty subsets of $\{x_0, \ldots, x_n\}$;
- $u_+ := \begin{cases} 0, & \text{if} \quad u \le 0; \\ u, & \text{if} \quad u > 0, \end{cases}$ the positive part of u;

LEMMA 1. *If $f_0, \ldots, f_n \in \mathcal{F}$ are linearly independent and $F_0, \ldots, F_n \in \mathcal{A}_n$ satisfy the condition*

$$d := \det(F_i(f_j))_{i,j=0}^{n} \neq 0, \tag{1}$$

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then, for any $A \in \mathcal{A}_n$*, the following formula is satisfied*

$$A = \sum_{k=0}^{n} \frac{(-1)^k}{d} \begin{vmatrix} A(f_0) & \cdots & A(f_n) \\ F_0(f_0) & \cdots & F_0(f_n) \\ \vdots & \ddots & \vdots \\ \not{F}_k(f_0) & \not{\cdots} & \not{F}_k(f_n) \\ \vdots & \ddots & \vdots \\ F_n(f_0) & \cdots & F_n(f_n) \end{vmatrix} \cdot F_k \tag{2}$$

(the notation $\not{F}_k$ *means that the k-th row is canceled).*

Proof. Let $f \in \mathcal{F}$. Taking into account the fact that f is a linear combination of f_0, …, f_n, it follows that

$$\begin{vmatrix} A(f) & A(f_0) & \cdots & A(f_n) \\ F_0(f) & F_0(f_0) & \cdots & F_0(f_n) \\ \vdots & \vdots & \ddots & \vdots \\ F_n(f) & F_n(f_0) & \cdots & F_n(f_n) \end{vmatrix} = 0.$$

We expand the determinant in terms of the first column and we take into consideration (1). □

Consider the polynomials

$$P_i(t) := \frac{\ell(t)}{(t - x_{i,0}) \dots (t - x_{i,n_i})}, \tag{3}$$

$i = 0$, …, n, where $\ell(t) := (t - x_0) \dots (t - x_n)$.

For y_0, …, $y_p \in \{x_0, …, x_n\}$, the reduction formula for divided differences gives

$$\Big[x_0, \dots, x_n; \tfrac{\ell(t)}{(t-y_0)\dots(t-y_p)} f(t)\Big]_t = [y_0, \dots, y_p;\ f]\,.$$

The the polynomials P_i satisfy

$$[x_0, \dots, x_n; P_i \cdot f] = [x_{i,0} \dots, x_{i,n_i};\ f]\,. \tag{4}$$

Lemma 2. *If* $Q \in \mathcal{P}_n$ *and*

$$[x_0, \dots, x_n; P\, Q] = 0, \qquad \forall P \in \mathcal{P}_n,$$

then $Q = 0$.

Proof. By virtue of (4), with $P_i(t) = \ell(t)/(t - x_i)$, $i = 0, \dots, n$, we obtain

$$Q(x_i) = [x_i;\ Q] = [x_0, \dots, x_n;\ P_i \cdot Q] = 0, \qquad i = 0, \dots, n,$$

hence $Q = 0$. □

2. MAIN RESULTS

THEOREM 3. *Any functional $A \in \mathcal{A}_n$ can be written in the form*

$$A = \sum_{k=0}^{n} \lambda_k \, [x_{k,0}, \ldots, x_{k,n_k}; \cdot\,], \tag{5}$$

for some $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$, if and only if *the polynomials* (3) *are linearly independent.*

Proof. Necessity. Let $P \in \mathcal{P}_n$ be an arbitrary polynomial of degree n. Consider the linear functional

$$A(f) = [x_0, \ldots, x_n;\, P \cdot f].$$

It follows that there exist $\lambda_k \in \mathbb{R}$, $k = 0, \ldots, n$, such that

$$A(f) = \sum_{k=0}^{n} \lambda_k \, [x_{k,0}, \ldots, x_{k,n_k}; f], \qquad \forall f \in \mathcal{F}.$$

By using (4), we obtain

$$A(f) = \sum_{k=0}^{n} \lambda_k \, [x_0, \ldots, x_n;\, P_k \cdot f] = [x_0, \ldots, x_n;\, \sum_{k=0}^{n} \lambda_k P_k \cdot f],$$

Consequently,

$$\Big[x_0, \ldots, x_n; \Big(P - \sum_{k=0}^{n} \lambda_k \, P_k\Big) f\Big] = 0, \qquad \forall f \in \mathcal{F},$$

therefore, by Lemma 2, we obtain

$$P = \sum_{k=0}^{n} \lambda_k P_k,$$

that is the set $\{P_0, \ldots, P_n\}$ is a basis in $\mathcal{P}_n$. It follows that the polynomials $P_0, \ldots, P_n$ are linearly independent.

Sufficiency. Suppose that the polynomials $P_0, \ldots, P_n$ are linearly independent and let $A \in \mathcal{A}_n$, $A(f) = \sum_{k=0}^{n} a_k f(x_k)$. There exist $\lambda_0, \ldots, \lambda_n \in \mathbb{R}$ such that

$$\sum_{k=0}^{n} a_k \tfrac{\ell(t)}{t - x_k} = \sum_{k=0}^{n} \lambda_k P_k(t), \qquad t \in \{x_0, \ldots, x_n\}.$$

It follows that

$$\sum_{k=0}^{n} a_k \tfrac{\ell(t)}{t - x_k} f(t) = \sum_{k=0}^{n} \lambda_k P_k(t) f(t), \qquad t \in \{x_0, \ldots, x_n\},$$

hence

$$\left[x_0, \ldots, x_n; \sum_{k=0}^{n} a_k \tfrac{\ell(t)}{t - x_k} f(t)\right]_t = \left[x_0, \ldots, x_n; \sum_{k=0}^{n} \lambda_k P_k(t) f(t)\right]_t,$$

therefore

$$\sum_{k=0}^{n} \left[x_0, \ldots, x_n; a_k \tfrac{\ell(t)}{t-x_k} f(t)\right]_t = \sum_{k=0}^{n} \left[x_0, \ldots, x_n; \lambda_k P_k(t) f(t)\right].$$

Consequently, by using (4), it follows that

$$\sum_{k=0}^{n} a_k f(x_k) = \sum_{k=0}^{n} \lambda_k \, [x_{k,0}, \ldots, x_{k,n_k}; f],$$

that is,

$$A(f) = \sum_{k=0}^{n} \lambda_k \, [x_{k,0}, \ldots, x_{k,n_k}; f].$$

□

In what follows we consider the functionals

$$D_k := [x_{k,0}, \ldots, x_{k,n_k}; \cdot], \qquad k = 0, \ldots, n. \tag{6}$$

LEMMA 4. *If* $\det(P_i(x_j))_{i,j=0}^n \neq 0$, *then* $\delta := \det(D_k(P_i))_{i,k=0}^n \neq 0$.

Proof. We consider the functionals $A_j \in \mathcal{A}_n$,

$$A_j(f) = f(x_j), \qquad j = 0, \ldots, n.$$

Since the polynomials (3) are linearly independent, by virtue of Theorem 3, it follows that there exists numbers λ_{jk} such that

$$A_j = \sum_{k=0}^{n} \lambda_{jk} \, D_k,$$

hence

$$P_i(x_j) = A_j(P_i) = \sum_{k=0}^{n} \lambda_{jk} \, D_k(P_i),$$

$i, j = 0, \ldots, n$. We get

$$\det(P_i(x_j))_{i,j=0}^n = \det(\lambda_{jk})_{k,j=0}^n \cdot \det(D_k(P_i))_{i,k=0}^n,$$

hence $\det(D_k(P_i))_{i,k=0}^n \neq 0$. □

In the next theorem we give a representation of λ_k from Eq. (5) in terms of the functionals (6).

THEOREM 5. *If* $\det(P_i(x_j))_{i,j=0}^n \neq 0$ *and* $A \in \mathcal{A}_n$, *then*

$$A = \sum_{k=0}^{n} \frac{(-1)^k}{\delta} \begin{vmatrix} A(P_0) & \cdots & A(P_n) \\ D_0(P_0) & \cdots & D_0(P_n) \\ \vdots & \ddots & \vdots \\ \not{D}_k(P_0) & \not{\cdots} & \not{D}_k(P_n) \\ \vdots & \ddots & \vdots \\ D_n(P_0) & \cdots & D_n(P_n) \end{vmatrix} [x_{k,0}, \ldots, x_{k,n_k}; \cdot].$$

Proof. We use Theorem 3, Lemma 1 and Lemma 4. □

3. APPLICATIONS

Corollary 6 (Popoviciu's Transformation Formula [4, Eq. (17)]). *If r is an integer, $0 \le r \le n$, then for all $A \in \mathcal{A}_n$ there exist numbers α_i such that*

$$A(f) = \sum_{0 \le i \le r-1} \alpha_i \, [x_0, \ldots, x_i; \, f] + \sum_{r \le i \le n} \alpha_i \, [x_{i-r}, \ldots, x_i; \, f].$$

Proof. In Theorem 3, we take

$$P_i(t) := \begin{cases} \frac{\ell(t)}{(t-x_0)\ldots(t-x_i)}, & 0 \le i \le r-1, \\ \frac{\ell(t)}{(t-x_{i-r})\ldots(t-x_i)}, & r \le i \le n. \end{cases}$$

□

Corollary 7. [2]. *If $a \le x_0 < \cdots < x_n \le b$ and $U : \mathcal{F}[a,b] \to \mathcal{F}[a,b]$ is a linear operator, then*

$$\begin{aligned} U(f; \, x) \;&=\; U(e_0; \, x) f(x_0) + U(e_1 - x_0 \, e_0; \, x) \, [x_0, x_1; f] \\ &\quad + \sum_{k=0}^{n-2} (x_{k+2} - x_k) U((\cdot - x_{k+1})_+; \, x) \; [x_k, x_{k+1}, x_{k+2} \, ; f] \, , \end{aligned}$$

for all $f \in \mathcal{F}[a,b]$, $x \in [a,b]$.

Proof. For fixed $x \in [a,b]$, we consider the functional $U(\cdot; \, x) \in \mathcal{A}_n$. The polynomials

$$\tfrac{\ell(t)}{t-x_0}, \; \tfrac{\ell(t)}{(t-x_0)(t-x_1)}, \; \tfrac{\ell(t)}{(t-x_0)(t-x_1)(t-x_2)}, \ldots, \; \tfrac{\ell(t)}{(t-x_{n-2})(t-x_{n-1})(t-x_n)},$$

are linearly independent. Therefore, by virtue of Theorem 3, it follows that there exist $\alpha(x)$, $\beta(x)$, $a_k(x) \in \mathbb{R}$, $k = 0, \ldots, n-2$, such that

$$U(f; \, x) = \alpha(x) \, [x_0; f] + \beta(x) \, [x_0, x_1; f] + \sum_{k=0}^{n-2} a_k(x) \; [x_k, x_{k+1}, x_{k+2} \, ; f] \, ,$$

for all $f \in \mathcal{F}[a,b]$.

Taking successively $f = e_0$ and $f = e_1$, we obtain

$$\alpha(x) = U(e_0; \, x)) \quad \text{and} \quad \beta(x) = U(e_1; \, x) - x_0 \, U(e_0; \, x),$$

hence

$$\begin{aligned} U(f; \, x) \;&=\; U(e_0, x) f(x_0) + (U(e_1; \, x) - x_0 \, U(e_0; \, x)) \, [x_0, x_1; f] \\ &\quad + \sum_{k=0}^{n-2} a_k(x) \; [x_k, x_{k+1}, x_{k+2} \, ; f] \, , \qquad \forall \, f \in \mathcal{F}[a,b]. \end{aligned}$$

In order to compute the numbers $a_k(x)$ we consider the functions

$$\varphi_i(t) := (t - x_{i+1})_+,$$

$i = 0, \ldots, n-2$. They satisfy the relations

$$[x_k, x_{k+1}, x_{k+2};\, \varphi_i] = \begin{cases} 0, & \text{if } k \neq i, \\ \frac{1}{x_{i+2}-x_i}, & \text{if } k = i, \end{cases}$$

$i, k = 0, \ldots, n-2$. From here, we deduce

$$U(\varphi_k;\, x) = \tfrac{a_k(x)}{x_{k+2}-x_k},$$

and the proof is completed. □

Corollary 8 (T. Popoviciu [6, p. 151]). *If $\Delta = \{x_0, \ldots, x_n\}$, then the broken line $S_\Delta(f)$ associated to f on Δ can be represented in the form*

$$\begin{aligned} S_\Delta(f)(x) &= f(x_0) + (x-x_0)\,[x_0, x_1; f] \\ &\quad + \sum_{k=0}^{n-2} (x_{k+2}-x_k)(x-x_{k+1})_+\ [x_k, x_{k+1}, x_{k+2}\,; f]\,, \qquad x \in \mathbb{R}. \end{aligned}$$

Proof. If in Corollary 7 we take $U := S_\Delta$ and use the fact that S_Δ preserves broken lines, i.e.,

$$\varphi_k(x) = S_\Delta(\varphi_k)(x),$$

the proof is completed. □

Corollary 9 (Newton Interpolating Formula). *If $x, x_0, \ldots, x_n$ are distinct points in $[a,b]$, then*

$$\begin{aligned} f(x) &= \sum_{k=0}^{n} (x-x_0)\ldots(x-x_{k-1})\,[x_0, \ldots, x_k; f] \\ &\quad + (x-x_0)\ldots(x-x_n)\,[x_0, \ldots, x_n, x; f], \end{aligned}$$

for all $f \in \mathcal{F}[a,b]$.

Proof. Let $\Delta = \{x_0, \ldots, x_{n+1}\}$ and $A \in \mathcal{A}_n$, $A(f) := f(x_{n+1})$. The polynomials

$$\tfrac{\ell(t)}{t-x_0},\ \ \tfrac{\ell(t)}{(t-x_0)(t-x_1)},\ \ldots,\ \tfrac{\ell(t)}{(t-x_0)(t-x_1)\cdots(t-x_{n+1})},$$

where $\ell(t) = (t-x_0)(t-x_1)\cdots(t-x_{n+1})$, are linearly independent. Therefore, by Theorem 3, it follows that there exist $\lambda_k \in \mathbb{R}$, $k = 0, \ldots, n+1$, such that

$$f(x_{n+1}) = \sum_{k=0}^{n+1} \lambda_k\ [x_0, \ldots, x_k; f]\,,$$

for all $f \in \mathcal{F}[a,b]$. In order to calculate the numbers λ_k we consider the functions

$$\varphi_0(t) := 1, \qquad \varphi_i(t) := (t-x_0)\ldots(t-x_{i-1}), \quad i = 1, \ldots, n+1.$$

We have

$$[x_0, \ldots, x_k;\, \varphi_i] = \delta_{ik}, \quad k, i = 0, \ldots, n+1.$$

We obtain

$$\lambda_i = \varphi_i(x_{n+1}), \qquad i = 0, \ldots, n+1.$$

Consequently,

$$f(x_{n+1}) = \sum_{k=0}^{n+1} \varphi_k(x_{n+1}) \, [x_0, \ldots, x_k; f] \,,$$

for all $f \in \mathcal{F}[a, b]$. With $x_{n+1} = x$ the proof is completed. □

REFERENCES

[1] DeVore, R. A. and Lorentz, G. G., *Constructive Approximation*, Springer Verlag, Berlin Heildelberg New York, 1993.

[2] Kacsó, D. P., *Approximation by means of piecewise linear functions*, Results Math., **35** (1–2), pp. 89–102, 1999.

[3] Popoviciu, E., *Mean value theorems and their connection to the interpolation theory*, Editura Dacia, Cluj, 1972 (in Romanian).

[4] Popoviciu, T., *Introduction à la théorie des différences divisées*, Bull. Math. de la Soc. Roumaine des Sci., **42** (1), pp. 65–78, 1940.

[5] Popoviciu, T., *Notes sur les fonctions convexes d'ordre supérieure (IX)*, Bull. Math. de la Soc. Roumaine des Sci., **43** (1–2), pp. 85–141, 1941.

[6] Popoviciu, T., *Curs de Analiză Matematică, Partea III-a, Continuitate*, Babeş-Bolyai University Publishing House, Cluj-Napoca, 1974 (in Romanian).

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