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ITERATED BOOLEAN SUMS OF BERNSTEIN AND RELATED OPERATORS

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Abstract. Let $(T(t))_{t\geq 0}$ be the semigroup associated with the classical Bernstein operators $(B_n)_{n\geq 1}$ on $C[0,1]$. We obtain rates of convergence for iterated Boolean sums of the operators $T\left(\frac{1}{n}\right)$.

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1. INTRODUCTION

Consider the classical Bernstein operators B_n on $C[0,1]$, defined by

$$B_n f(x) = \sum_{k=0}^{n} \tbinom{n}{k} x^k (1-x)^{n-k}, \quad f \in C[0,1],\ x \in [0,1].$$

The associated semigroup $(T(t))_{t\geq 0}$ can be described by

$$T(t)f = \lim_{n\to\infty} B_n^{[nt]} f \text{ uniformly on } [0,1],$$

where $t \geq 0$ and $f \in C[0,1]$; see [2, Sect. 6.3], [8].

If $P, Q : X \longrightarrow X$ are linear operators on an arbitrary linear space X, their Boolean sum is defined to be

$$P \oplus Q := P + Q - PQ.$$

Considering the k-fold Boolean sum, define

$$B_{n,k} := B_n \oplus \cdots \oplus B_n = I - (I - B_n)^k,$$

where I is the identity operator on $C[0,1]$.

The operators $B_{n,k}$ were introduced independently by Micchelli [6], Mastroianni and Occorsio [5], and Felbecker [3]. They were further investigated in several papers; for surveys, references and historical remarks see [4], [9, Chapter 26].

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In this paper we consider the k-fold Boolean sum of $T\left(\frac{1}{n}\right)$:

$$T_{n,k} := T\left(\tfrac{1}{n}\right) \oplus \cdots \oplus T\left(\tfrac{1}{n}\right) = I - \left(I - T\left(\tfrac{1}{n}\right)\right)^k$$

and obtain rates of convergence when k is fixed and $n \to \infty$.

2. MAIN RESULTS

THEOREM 2.1. *For all $f \in C[0,1]$ and $n \geq 1$,*

$$\left\|T\left(\tfrac{1}{n}\right) f - f\right\| \leq \tfrac{5}{4}\omega(f, \sqrt{1-\mathrm{e}^{-1/n}}), \tag{1}$$

where $\|\cdot\|$ is the uniform norm and ω is the usual modulus of continuity.

Proof. Let $e_j(x) = x^j,\ x \in [0,1],\ j = 0,1,2.$
Then (see, e.g., [8]):

$$T\left(\tfrac{1}{n}\right) e_0 = e_0, \tag{2}$$

$$T\left(\tfrac{1}{n}\right) e_1 = e_1, \tag{3}$$

$$T\left(\tfrac{1}{n}\right) e_2 = (1-\mathrm{e}^{-1/n})e_1 + \mathrm{e}^{-1/n}e_2. \tag{4}$$

Consequently, from [2, Prop.5.1.5] we get

$$\left|T\left(\tfrac{1}{n}\right) f(x) - f(x)\right| \leq \left(1 + \tfrac{1}{\delta^2}\left(T\left(\tfrac{1}{n}\right) e_2(x) - e_2(x)\right)\right)\omega(f,\delta)$$

for all $\delta > 0$.
Using (4) and choosing $\delta := \sqrt{1-\mathrm{e}^{-1/n}}$ we obtain for $x \in [0,1]$

$$\left|T\left(\tfrac{1}{n}\right) f(x) - f(x)\right| \leq (1 + x(1-x))\omega(f, \sqrt{1-\mathrm{e}^{-1/n}}). \tag{5}$$

Now, (1) is a consequence of (5). □

REMARK 2.1. For $f \in C^2[0,1],\ x \in [0,1]$ and $n \geq 1$ we have also (see [8]):

$$\left|T\left(\tfrac{1}{n}\right) f(x) - f(x)\right| \leq (1-\mathrm{e}^{-1/n})\tfrac{x(1-x)}{2}\|f''\|. \tag{6}$$

THEOREM 2.2. *Let $f \in C[0,1]$, $n \geq 1$ and $k \geq 1$. Then:*

$$\|T_{n,k}f - f\| \leq 5 \cdot 2^{k-3}\omega(f, \sqrt{1-\mathrm{e}^{-1/n}}). \tag{7}$$

Proof. Let us remark that

$$\begin{aligned} T_{n,k} - I &= -\left(I - T\left(\tfrac{1}{n}\right)\right)^k \\ &= \left(T\left(\tfrac{1}{n}\right) - I\right)\left(I - T\left(\tfrac{1}{n}\right)\right)^{k-1} \\ &= \left(T\left(\tfrac{1}{n}\right) - I\right)\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}\left(T\left(\tfrac{1}{n}\right)\right)^j. \end{aligned}$$

Since

$$\left(T\left(\tfrac{1}{n}\right)\right)^{j} = T\left(\tfrac{j}{n}\right),$$

we get

$$T_{n,k} - I = \sum_{j=0}^{k-1}(-1)^{j}\tbinom{k-1}{j}\left(T\left(\tfrac{j+1}{n}\right) - T\left(\tfrac{j}{n}\right)\right). \tag{8}$$

This yields

$$\begin{aligned}\|T_{n,k}f - f\| &\leq \sum_{j=0}^{k-1}\tbinom{k-1}{j}\Big\|T\left(\tfrac{j}{n}\right)\left(T\left(\tfrac{1}{n}\right)f - f\right)\Big\| \\ &\leq \sum_{j=0}^{k-1}\tbinom{k-1}{j}\Big\|T\left(\tfrac{j}{n}\right)\Big\|\Big\|T\left(\tfrac{1}{n}\right)f - f\Big\|.\end{aligned} \tag{9}$$

On the other hand, $T(\frac{j}{n})$ is a positive linear operator, and $T(\frac{j}{n})e_0 = e_0$; it follows that $\|T(\frac{j}{n})\| = 1$.

Now from (9) and (1) we infer

$$\|T_{n,k}f - f\| \leq \tfrac{5}{4}\omega(f, \sqrt{1 - \mathrm{e}^{-1/n}}) \cdot 2^{k-1}$$

and this entails (7). □

Remark 2.2. The estimates

$$\|B_{n,k}f - f\| \leq \tfrac{3}{2}(2^{k} - 1)\omega(f, n^{-1/2}), \tag{10}$$

$$\|B_{n,k}f - f\| \leq \left(2^{k} - 1 + \tfrac{n}{4}\left(2^{k} - \left(2 - \tfrac{1}{n}\right)^{k}\right)\right)\omega(f, n^{-1/2}) \tag{11}$$

were obtained by Micchelli in [6], respectively by Agrawal and Kasana in [1].

They were improved by Biancamaria Della Vecchia and the author (see [9, Chapter 26]):

$$\|B_{n,k}f - f\| \leq 5 \cdot 2^{k-3}\omega(f, n^{-1/2}). \tag{12}$$

In the sequel $[a, b, c; f]$ will denote the divided difference of the function f at the points $a < b < c$.

Theorem 2.3. *Let $n, k \geq 1$, $f \in C[0,1]$ and $x \in [0,1]$ be given. Then there exist $0 \leq a < b < c \leq 1$ and $0 \leq p < q < r \leq 1$ such that:*

$$\begin{aligned}&T_{n,k}f(x) - f(x) = (1 - \mathrm{e}^{-1/n})\tfrac{x(1-x)}{2}\times \\ &\times\Big(((1 + \mathrm{e}^{-1/n})^{k-1} + (1 - \mathrm{e}^{-1/n})^{k-1})[a, b, c; f] - \\ &-((1 + \mathrm{e}^{-1/n})^{k-1} - (1 - \mathrm{e}^{-1/n})^{k-1})[p, q, r; f]\Big).\end{aligned} \tag{13}$$

Proof. From (8) we get

$$T_{n,k}f(x) - f(x) = \mu(f) - \nu(f), \tag{14}$$

where

$$\mu(f) := \sum_i \binom{k-1}{2i} \left(T\left(\tfrac{2i+1}{n}\right) f(x) - T\left(\tfrac{2i}{n}\right) f(x) \right),$$
$$\nu(f) := \sum_i \binom{k-1}{2i-1} \left(T\left(\tfrac{2i}{n}\right) f(x) - T\left(\tfrac{2i-1}{n}\right) f(x) \right).$$

Using $T(t)e_2(x) = (1-\mathrm{e}^{-t})x + \mathrm{e}^{-t}x^2$ (see[8]) we deduce

$$\mu(e_2) = (1-\mathrm{e}^{-1/n})x(1-x) \sum_i \binom{k-1}{2i} \mathrm{e}^{-2i/n},$$
$$\nu(e_2) = (1-\mathrm{e}^{-1/n})x(1-x) \sum_i \binom{k-1}{2i-1} \mathrm{e}^{-(2i-1)/n},$$

and then

$$\mu(e_2) = (1-\mathrm{e}^{-1/n})\tfrac{x(1-x)}{2} \left((1+\mathrm{e}^{-1/n})^{k-1} + (1-\mathrm{e}^{-1/n})^{k-1} \right), \tag{15}$$

$$\nu(e_2) = (1-\mathrm{e}^{-1/n})\tfrac{x(1-x)}{2} \left((1+\mathrm{e}^{-1/n})^{k-1} - (1-\mathrm{e}^{-1/n})^{k-1} \right). \tag{16}$$

Let $g \in C[0,1]$ be convex and $0 \le s \le t$. Then $T(t-s)g \ge g$ (see [2, Cor. 6.3.8]) and consequently $T(t)g = T(s)T(t-s)g \ge T(s)g$. It follows that $\mu(g) \ge 0$ and $\nu(g) \ge 0$.

By a classical result of Tiberiu Popoviciu (see [7]) there exist $0 \le a < b < c \le 1$ and $0 \le p < q < r \le 1$ such that

$$\mu(f) = \mu(e_2)[a,b,c;f], \tag{17}$$

$$\nu(f) = \mu(e_2)[p,q,r;f]. \tag{18}$$

Now, (13) is a consequence of (14)–(18). □

Corollary 2.4. *Let $n,k \ge 1$, $x \in [0,1]$, $f \in C^2[0,1]$. Then*

$$|T_{n,k}f(x) - f(x)| \le (1-\mathrm{e}^{-1/n})\tfrac{x(1-x)}{2}(1+\mathrm{e}^{-1/n})^{k-1}\|f''\|. \tag{19}$$

Proof. By using the mean-value property of the divided differences we obtain

$$|[a,b,c;f]| \le \tfrac{1}{2}\|f''\|, \quad 0 \le a < b < c \le 1.$$

Now, (19) follows from (13). □

REFERENCES

[1] Agrawal, P.N. and Kasana, H.S., *On the iterative combinations of Bernstein polynomials,* Demonstratio Math., **17**, pp. 777–783, 1984.

[2] Altomare, F. and Campiti, M., *Korovkin-type Approximation Theory and its Applications,* Walter de Gruyter, Berlin - New York, 1994.

[3] Felbecker, G., *Linearkombinationen von iterierten Bernsteinoperatoren,* Manuscripta Math., **29**, pp. 229–248, 1979.

[4] Gonska, H.H. and Zhou, X.L., *Approximation theorems for the iterated Boolean sums of Bernstein operators,* J. Comp. Appl. Math., **53**, pp. 21–31, 1994.

[5] Mastroianni, G. and Occorsio, M.R., *Una generalizzazione dell'operatore di Bernstein,* Rend. Accad. Sci. Mat. Fis. Nat. Napoli, **44**, pp. 151–169, 1977.

[6] Micchelli, C.A., *The saturation class and iterates of the Bernstein polynomials,* J. Approximation Theory, **8**, pp. 1–18, 1973.

[7] Rasa, I., *Sur les fonctionnelles de la forme simple au sens de T. Popoviciu,* Rev. Anal. Numér. Théor. Approx., **9**, pp. 261–268, 1980.

[8] Rasa, I., *Estimates for the semigroup associated with Bernstein operators,* Rev. Anal. Numér. Théor. Approx., **33**, 243–245, 2004.

[9] Vladislav, T. and Rasa, I., *Analiză Numerică. Aproximare, problema lui Cauchy abstractă, proiectori Altomare,* Editura Tehnică, Bucureşti, 1999.

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