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BILATERAL APPROXIMATIONS OF THE ROOTS OF SCALAR EQUATIONS BY LAGRANGE-AITKEN-STEFFENSEN METHOD OF ORDER THREE*

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Abstract. We study the monotone convergence of two general methods of Aitken-Steffenssen type. These methods are obtained from the Lagrange inverse interpolation polynomial of degree two, having controlled nodes. The obtained results provide information on controlling the errors at each iteration step.

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Keywords. Aitken-Steffenssen methods, Lagrange inverse interpolation.

1. INTRODUCTION

It is well known that the Steffensen, Aitken, and Aitken-Steffensen methods are obtained from the inverse Lagrange interpolation polynomial of degree one, with controlled nodes [8]–[12], [6]. Consider the equation

$$f(x) = 0 \tag{1.1}$$

where

$$f : [a, b] \to R,\ \ a, b \in R,\ \ a < b.$$

We also consider the following three equations, each of them equivalent with equation (1.1):

$$x = g(x),\ \ g : [a, b] \to [a, b] \tag{1.2}$$

and

$$\begin{aligned} x &= g_1(x),\ \ g_1 : [a, b] \to [a, b], \\ x &= g_2(x),\ \ g_2 : [a, b] \to [a, b]. \end{aligned} \tag{1.3}$$

The Steffensen method in given by relations

$$x_{n+1} = x_n - \frac{f(x_n)}{[x_n, g(x_n); f]},\quad x_0 \in [a, b],\ \ n = 0, 1, 2, \ldots, \tag{1.4}$$

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and, analogously, the Aitken method is of the following form:

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(x_n); f]}, \quad x_0 \in [a,b],\ n = 0,1,2,\ldots. \tag{1.5}$$

Finally, the Aitken-Steffensen method is given by the relations:

$$x_{n+1} = g_1(x_n) - \frac{f(g_1(x_n))}{[g_1(x_n), g_2(g_1(x_n)); f]}, \quad x_0 \in [a,b],\ n = 0,1,2,\ldots. \tag{1.6}$$

The order of convergence of all three methods (1.4)–(1.6) is at least two, and this order depends on the functions g and g_1, g_2 respectively. Essentially, the methods (1.4)–(1.6) are obtained from the method of chord where the interpolation nodes depend on the functions g and respectively g_1, g_2. In papers [1], [8], [9], [12] some conditions had to be considered in order that all the three methods (1.4)–(1.6) generate two sequences $(u_n)_{n\geq 0}$ and $(v_n)_{n\geq 0}$ with the properties:

α) *The sequence* $(u_n)_{n\geq 0}$ *is increasing and the sequence* $(v_n)_{n\geq 1}$ *is decreasing:*

β) $\lim u_n = \lim v_n = \bar{x}$, *where* $\bar{x}$ *is the root of equation* (1.1), $\bar{x} \in [a,b]$.

Practically, such sequences are very interesting, because by inequalities

$$\max\{\bar{x} - u_n, v_n - \bar{x}\} \leq v_n - u_n, \quad n = 0,1,2,\ldots,$$

the errors of approximation at every step of iteration may be controlled.

Let $a_1, a_2, a_3 \in [a,b]$ be three nodes of interpolation, and let b_1, b_2, b_3 the values of the function f, i.e.

$$b_1 = f(a_1), \quad b_2 = f(a_2), \quad b_3 = f(a_3).$$

Suppose that the function $f : [a,b] \to F$, is bijective where $F = f([a,b])$.

Then there exists $f^{-1} : F \to [a,b]$ and the following equality holds [12], [14]:

$$\begin{aligned} f^{-1}(y) &= a_1 + [b_1, b_2; f^{-1}](y - b_1) \\ &\quad + [b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2) \\ &\quad + [y, b_1, b_2, b_3; f^{-1}](y - b_1)(y - b_2)(y - b_3), \end{aligned} \tag{1.7}$$

for every $y \in F$.

If $\bar{x} \in [a,b]$ is the root of equation (1.1), then $\bar{x} = f^{-1}(0)$, and by (1.7) one obtains the following representation of $\bar{x}$:

$$\begin{aligned} \bar{x} &= a_1 - [b_1, b_2; f^{-1}]b_1 + [b_1, b_2, b_3; f^{-1}]b_1 b_2 \\ &\quad - [0, b_1, b_2, b_3; f^{-1}]b_1 b_2 b_3. \end{aligned} \tag{1.8}$$

By relations (see [12])

$$[b_1, b_2; f^{-1}] = \frac{1}{[a_1, a_2; f]}$$

and

$$[b_1, b_2, b_3; f^{-1}] = -\frac{[a_1, a_2, a_3; f]}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]},$$

using (1.8), one obtains the following approximation for $\bar{x}$:

$$a_4 = a_1 - \frac{f(a_1)}{[a_1, a_2; f]} - \frac{[a_1, a_2, a_3; f]f(a_1)f(a_2)}{[a_1, a_2; f][a_1, a_3; f][a_2, a_3; f]} \tag{1.9}$$

and the error:

$$\bar{x} - a_4 = -[0, b_1, b_2, b_3; f^{-1}]f(a_1)\, f(a_2) f(a_3)\,. \tag{1.10}$$

If $f \in C^3([a, b])$ and $f'(x) \neq 0$, $\forall x \in [a, b]$, then $f^{-1} \in C^3(F)$ and the following equality holds (see [12], [16]):

$$[f^{-1}(y)]''' = \frac{3[f''(x)]^2 - f'(x)f'''(x)}{[f'(x)]^5}, \tag{1.11}$$

where $y = f(x)$.

Using the mean value formula for divided differences (see [12]) it follows that there exists $\eta \in F$ such that

$$[0, b_1, b_2, b_3; f^{-1}] = \frac{(f^{-1})'''(\eta)}{6}. \tag{1.12}$$

Because f is bijective, it follows that there exists $\xi \in [a, b]$ such that $\eta = f(\xi)$, and by (1.11) and (1.12) one obtains:

$$[0, b_1, b_2, b_3; f^{-1}] = \frac{3[f''(\xi)]^2 - f'(\xi)f'''(\xi)}{6[f'(\xi)]^5}. \tag{1.13}$$

By (1.9), if one considers particular nodes a_1, a_2, a_3 it is possible to obtain different methods of Steffensen type, of Aitken type or of Aitken-Steffensen type.

Let $x_n \in [a, b]$ be an approximation of the root $\bar{x}$ of equation (1.1).

If one considers $a_1 = x_n$, $a_2 = g(x_n)$, $a_3 = g(g(x_n))$, then it follows the following method of Steffensen type [8], [9], [12]:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{[x_n, g(x_n); f]} \\ &\quad - \frac{[x_n, g(x_n), g(g(x_n)); f]f(x_n)f(g(x_n))}{[x_n, g(x_n); f][x_n, g(g(x_n)); f][g(x_n), g(g(x_n)); f]}, \end{aligned} \tag{1.14}$$

$x_0 \in [a, b]$, $n = 0, 1, 2, \ldots$.

If $a_1 = x_n$, $a_2 = g_1(x_n)$, $a_3 = g_2(g_1(x_n))$, then one obtains the following method of Aitken-Steffensen type:

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{(x_n, g_1(x_n); f]} \\ &\quad - \frac{[x_n, g_1(x_n), g_2(g_1(x_n)); f]f(x_n)f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(g_1(x_n)); f][g_1(x_n), g_2(g_1(x_n)); f]}, \end{aligned} \tag{1.15}$$

$n = 0, 1, \ldots, x_0 \in [a, b]$, and finally, for $a_1 = x_n,\ a_2 = g_1(x_n)$, $a_3 = g_2(x_n)$ one obtains the following method of Aitken type:

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n)}{[x_n, g_1(x_n); f]} \\
&\quad - \frac{[x_n, g_1(x_n), g_2(x_n); f] f(x_n) f(g_1(x_n))}{[x_n, g_1(x_n); f][x_n, g_2(x_n); f][g_1(x_n), g_2(x_n); f]}, \\
n &= 0, 1, \ldots,\ x_0 \in [a, b].
\end{aligned}
\tag{1.16}
$$

Using the symmetry of Lagrange polynomial with respect to nodes, by permutations of a_1, a_2, a_3 in methods (1.14)–(1.16), one obtains the same results for x_{n+1}.

In [14] conditions are given in order that method (1.14) generates sequences approximating the root of equation (1.1) bilaterally. In this paper we study methods (1.15) and (1.16) and we obtain same conditions in order that the sequences generated by there methods are bilateral approximations of the root $\bar{x}$ of equations (1.1).

2. THE CONVERGENCE OF AITKEN-STEFFENSEN METHOD

In the following we study the method (1.15) and we search the conditions on the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(g_1(x_n))_{n\geq 0}$ generated from this method in order that they are monotonic sequences, bilaterally approximating the root $\bar{x}$ of equation (1.1).

We need the following hypothesis:

a) *g_1 is increasing on* $[a, b]$;
b) *g_2 is continuous and decreasing on* $[a, b]$;
c) *equation* (1.1) *has a solution* $\bar{x} \in [a, b]$ *and* $g_1(\bar{x}) = g_2(\bar{x}) = \bar{x}$,
d) *function f is in* $C^3([a, b])$, *and for every* $x \in [a, b]$ *the following relation is fulfilled:*

$$3[f''(x)]^2 - f'(x)f'''(x) < 0; \tag{2.1}$$

e) *function g_1 satisfies inequality*

$$|g_1(x) - g_1(\bar{x})| \leq L\,|x - \bar{x}|\,, \forall x \in [a, b],$$

where $0 < L < 1$.

Concerning the convergence of sequence $(x_n)_{n\geq 0}$ generated by (1.15), the following Theorem holds:

THEOREM 2.1. *Let $x_0 \in [a, b]$ and $f,\ g_1,\ g_2$ verify the following conditions:*

i$_1$) *f is increasing on* $[a, b]$;
ii$_1$) *f is convex on* $[a, b]$;
iii$_1$) *functions g_1, g_2 and f verify hypotheses* $a) - e)$;
iv$_1$) $x_0 > \bar{x}$ *and* $g_2(g_1(x_0)) \geq a$.

Then the sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ *and* $(g_2(g_1(x_n))_{n\geq 0}$ *generated by* (1.15) *verify the properties*:

j$_1$) *sequences* $(x_n)_{n\geq 0}$ *and* $(g_1(x_n))_{n\geq 0}$ *are decreasing and bounded from below by* $\bar{x}$;

jj$_1$) *sequence* $(g_2(g_1(x_n))_{n\geq 0}$ *is increasing and bounded from above by* $\bar{x}$;

jjj$_1$) *at every iteration step the following inequalities hold*:

$$x_n - \bar{x} \leq x_n - g_2(g_1(x_n)),\ n = 0, 1, \ldots; \tag{2.2}$$

jv$_1$) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n) = \bar{x}$.

Proof. By hypotheses a) and $x_0 > \bar{x}$ it follows $g_1(x_0) > g_1(\bar{x})$, i.e $g_1(x_0) > \bar{x}$. Hypothesis e) implies $g_1(x_0) < x_0$. From **iv**$_1$) it follows $g_2(g_1(x_0) \geq a$ and from b) and $g_1(x_0) > \bar{x}$ one obtains $g_2(g_1(x_1)) < \bar{x}$. Now using **i**$_1$) it follows:

$$f(x_0) > 0, \quad f(g_1(x_0)) > 0$$

and

$$f(g_2(g_1(x_0)) < 0,$$

and by considering **i**$_1$) and **ii**$_1$) for $n = 0$ in (1.15) it follows that $x_1 < x_0$. Using (1.13), by hypotheses d) and (1.10) for $a_4 = x_1$ and $b_1 = f(x_0)$, $b_2 = f(g_1(x_0))$, $b_3 = f(g_2(g_1(x_0))$ it follows $\bar{x} - x_1 < 0$, i.e $x_1 > \bar{x}$. By hypotheses a) and $x_1 < x_0$ it follows $g_1(x_1) < g_1(x_0)$ and then, by b) one obtains $g_2(g_1(x_1)) \geq g_2(g_1(x_0))$, and $g_2(g_1(x_1)) < \bar{x}$. Let x_m, $m \in \mathbb{N}$ be an element of sequence $(x_n)_{n\geq 0}$ generated by (1.15) and suppose that $x_m > \bar{x}$. Then one obtains the relations:

$$g_2(g_1(x_m)) < g_2(g_1(x_{m+1})) < \bar{x} < g_1(x_{m+1}) < x_{m+1} < g_1(x_m) < x_m. \tag{2.3}$$

Inequality $x_{m+1} < g_1(x_m)$ follows from equality:

$$\begin{aligned} x_{m+1} &= x_m - \frac{f(x_m)}{[x_m, g_1(x_m); f]} \\ &\quad - \frac{[x_m, g_1(x_m), g_2(g_1(x_m)); f]f(x_m)f(g_1(x_m)}{[x_m, g_1(x_m); f][x_m, g_2(g_1(x_m)); f][g_1(x_m), g_2(g_1(x_m)); f]} \\ &= g_1(x_m) - \frac{f(g_1(x_m)}{[x_m, g_1(x_m); f]} \\ &\quad - \frac{[x_m, g_1(x_m), g_2(g_1(x_m)); f]f(g_1(x_m))f(x_m)}{[x_m, g_1(x_m); f][x_m, g_2(g_1(x_m)); f][g_1(x_m), g_2(g_1(x_m)); f]} \end{aligned}$$

and from the hypothesis of the theorem.

From relations (2.3) it follows (2.2) and conclusions **j**$_1$) and **jj**$_1$). Conclusion **jv**$_1$) is obvious. The theorem is proved. □

Remark 2.2. If function f is concave and decreasing on $[a, b]$, then function $h : [a, b] \to \mathbb{R}$ defined by $h(x) = -f(x)$ in convex and increasing. The relation (2.1) is also verified for h. Consequently the sequence generated by (1.15) for function h, verifies all the conditions of Theorem 2.1, and then the conclusions of this theorem hold.

An analogous proof with that from Theorem 2.1 is valid for the following:

THEOREM 2.3. *Let $x_0 \in [a,b]$ and f, g_1, g_2 verify the following conditions:*

i$_2$) *function f is increasing on $[a,b]$;*
ii$_2$) *function f is concave on $[a,b]$;*
iii$_2$) *functions g_1, g_2 and f verify the hypotheses $a) - e)$;*
iv$_2$) $x_0 < \bar{x}$ *and* $g_2(g_1(x_0) \leq b$.

Then sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(g_1(x_{n\geq 0})))$ generated by (1.15) *have the following properties*:

j$_2$) *sequences $(x_n)_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ are increasing and bounded from above by $\bar{x}$;*
jj$_2$) *sequence $g_2(g_1(x_n))$ is decreasing and bounded from below by $\bar{x}$;*
jjj$_2$) *the following relations hold:*

$$\bar{x} - x_n \leq g_2(g_1(x_n)) - x_n,\ n = 0, 1, 2, \ldots, \tag{2.4}$$

jv$_2$) $\lim x_n = \lim g_1(x_n) = \lim g_2(g_1(x_n)) = \bar{x}$.

REMARK 2.4. If function f is decreasing and convex, then $h_1 : [a,b] \to \mathbb{R}$ defined by $h_1(x) = -f(x)$ is increasing and concave, $h_1$verifies all hypotheses of Theorem 2.3 and consequently all the conclusions **j**$_2$) $-$ **jv**$_2$) hold.

3. CONVERGENCE OF AITKEN TYPE METHOD

In order to study the convergence of the sequences generated by (1.16) we must suppose that functions f, g_1 and g_2 verify hypotheses a)-e) of section 2.

The following theorem holds:

THEOREM 3.1. *Let $x_0 \in [a,b]$ and the functions f, g_1 and g_2 verify the conditions:*

i$_3$) *f is increasing an $[a,b]$;*
ii$_3$) *f is convex an $[a,b]$;*
iii$_3$) *f, g_1 and g_2 verifies the hypotheses $a) - e)$;*
iv$_3$) $x_0 > \bar{x}$ *and* $g_2(x_0) > a$.

Then sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(x_n))_{n\geq 0}$ generated by (1.16) *have the properties:*

j$_3$) *sequences $(x_n)_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ are decreasing and bounded from below by $\bar{x}$;*
jj$_3$) *sequence $(g_2(x_n))_{n\geq 0}$ is increasing and bounded from above by $\bar{x}$;*
jjj$_3$) *the following relations hold,*

$$x_n - \bar{x} \leq x_n - g_2(x_n),\ n = 0, 1, \ldots;$$

jv$_3$) $\lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = \bar{x}$.

Proof. Let be $x_m \in [a,b]$, $x_m > \bar{x}$ and $g_2(x_m) > a$, where $m \in \mathbb{N}$. By a) it follows that $g_1(x_m) > \bar{x}$ and by b) $g_2(x_m) < \bar{x}$. Using e) one obtains $g_1(x_m) < x_m$. By hypothesis $\mathbf{i}_3$) and $\mathbf{ii}_3$), and by the above relations and (1.16), for $n = m$ it follows $x_{m+1} < x_m$.

For $a_1 = x_m$, $a_2 = g_1(x_m)$ and $a_3 = g_2(x_m)$ in (1.10), by (1.13) and hypothesis d) it follows that $x_{m+1} > \bar{x}$. Observe that x_{m+1} in (1.16) may by expressed in the following way:

$$\begin{aligned} x_{m+1} &= g_1(x_m) - \frac{f\left(g_1(x_m)\right)}{[g_1(x_m), x_m; f]} - \\ &- \frac{[x_m, g_1(x_m), g_2(x_m); f] f(g_1(x_m)) f(x_m)}{[x_m, g_1(x_m); f][x_m, g_2(x_m); f][g_1(x_m), g_2(x_m); f]} \end{aligned}$$

and then $x_{m+1} < g_1(x_m)$. By relation $x_{m+1} < x_m$ it follows that $g_2(x_{m+1}) > g_2(x_m)$. Consequently it follows that for every $m \in \mathbb{N}$, the following relations hold:

$$g_2(x_m) < g_2(x_{m+1}) < \bar{x} < g_1(x_{m+1}) < x_{m+1} < g_1(x_m) < x_m.$$

By these relations conclusions $\mathbf{jjj}_3$) and $\mathbf{jv}_3$) also follow. □

REMARK 3.2. If function f is decreasing and concave, then function $h(x) = -f(x)$, $x \in [a,b]$ verifies all the hypothesis of Theorem 3.1 and consequently the sequence $(x_n)_{n\geq 1}$ generated by (1.16) verifies all the conclusions in Theorem 3.1.

Analogously, the following result can be proved

THEOREM 3.3. *Let $x_0 \in [a,b]$ and functions f, g_1, g_2 have the following properties:*

- $\mathrm{i}_{4)}$ *f is increasing on $[a,b]$;*
- ii_4) *f is concave on $[a,b]$;*
- iii_4) *f, g_1, and g_2 verify hypothesis* a)-e);
- iv_4) *$x_0 < \bar{x}$ and $g_2(x_0) \leq b$.*

Then sequences $(x_n)_{n\geq 0}$, $(g_1(x_n))_{n\geq 0}$ and $(g_2(x_n))_{n\geq 0}$ generated by (1.16) *verify the properties:*

- j_4) *$(x_n)_{n\geq 0}$ and $(g_1(x_n))_{n\geq 0}$ are increasing and bounded from above by $\bar{x}$;*
- jj_4) *sequence $(g_2(x_n))_{n\geq 0}$ is decreasing and bounded from below by $\bar{x}$;*
- jjj_4) *the following relations hold*:

$$\bar{x} - x_n \leq g_2(x_n) - x_n, n = 0, 1, 2, \ldots,$$

- jv_4) $\lim x_n = \lim g_1(x_n) = \lim g_2(x_n) = \bar{x}$.

REMARK 3.4. If function f is decreasing and convex, then function $h(x) = -f(x)$, $x \in [a,b]$ is increasing and concave. It follows that sequence $(x_n)_{n\geq 0}$ generated by (1.16) verifies the hypotheses and all the conclusions of Theorem 3.3.

4. THE DETERMINATION OF THE AUXILIARY FUNCTIONS

In the following, for every situation concerning the monotonicity and convexity of function f, functions g_1 and g_2 can be determined such that conditions a), b), c) and e) should be verified. In the following we thoroughly present the case in which function f is increasing and convex.

Supposing that $f'(x) > 0$ for every $x \in [a, b]$, one considers the functions:

$$g_1(x) = x - \tfrac{f(x)}{f'(b)}, \tag{4.1}$$

and

$$g_2(x) = x - \tfrac{f(x)}{f'(a)}. \tag{4.2}$$

Then

$$g_1'(x) = 1 - \tfrac{f'(x)}{f'(b)} \geq 0$$

for every $x \in [a, b]$ and, consequently g_1 is increasing. Analogously

$$g_2'(x) = 1 - \tfrac{f'(x)}{f'(a)} \leq 0$$

for every $x \in [a, b]$ and then g_2 is a decreasing function. It follows that g_1 and g_2 verify hypotheses a) and b). The hypothesis e) is also verified because $g_1'(x) < 1$, for every $x \in [a, b]$. In iv_1) the inequality $g_2(g_1(x_0) \geq a$ holds. Because $g_1(x_0) < x_0$ and $x_0 > \bar{x}$, the above inequality is verified if $g_2(x_0) \geq a$. This follows, because $g_2(g_1(x_0)) > g_2(x_0)$ (g_2 is a decreasing functions). This means that the following relation must hold:

$$x_0 - \tfrac{f(x_0)}{f'(a)} \geq a$$

i.e.

$$x_0 \geq a + \tfrac{f(x_0))}{f'(a)}.$$

Because $f(x_0) > 0$ and $f'(a) > 0$, for the veridicity of last inequality it is sufficient that

$$a + \tfrac{f(x_0)}{f'(a)} \leq \bar{x},$$

where $\bar{x}$ is the root of equation (1.1). This last inequality may be realized if x_0 is sufficiently close to $\bar{x}$.

Consequently, the hypotheses of theorems 2.1 and 3.1 are realized for g_1 and g_2 considered above. The other cases may be similarly analyzed.

5. ORDER OF CONVERGENCE

In the following we prove that every method (1.15) and (1.16) have the order of convergence three. The order of convergence for method (1.14) was treated in [14], and this order is at least three. Assume the following hypotheses:

α) function g_2 verifies relation

$$|g_2(x) - g_2(\bar{x})| \leq p\,|x - \bar{x}|\,,$$

for every $x \in [a,b]$, where $p > 0$, $p \in \mathbb{R}$;

β) $m \leq |f'(x)| \leq M$, for every $x \in [a,b]$, where $m > 0$ and $M > 0$ are real numbers.

γ) $\left|3[f'(x)]^2 - f'(x)f'''(x)\right| \leq q$, for every $x \in [a,b]$, where $q > 0$, $q \in \mathbb{R}$.

In hypotheses α), β), γ) and e), by (1.10) and (1.13), for sequence $(x_n)_{n\geq 0}$ generated by (1.15), one obtains:

$$|\bar{x} - x_{n+1}| \leq \tfrac{qM^3L^2p}{6m^5}\,|\bar{x} - x_n|^3\,, n = 0,1,2,\ldots$$

and this means that the order of convergence for method (1.15) is at least three.

With the same hypotheses, for tree sequence $(x_n)_{n\geq 0}$ generated by (1.16) it follows:

$$|\bar{x} - x_{n+1}| \leq \tfrac{qM^3Lp}{6m^5}\,|\bar{x} - x_n|^3\,,\ n = 0,1,2,\ldots$$

i.e. the order of convergence of (1.16) is also at least three.

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