

## On the chord method

In the paper [1], I.K. Argyros considers as divided difference of the mapping  $f : X_1 \rightarrow X_2$ , where  $X_1$  and  $X_2$  are Banach spaces, a linear mapping  $[x, y; f] \in \mathcal{L}(X_1, X_2)$  which fulfils the following conditions:

- (a)  $[x, y; f](y - x) = f(y) - f(x)$ , for every  $x, y \in D$ , where  $D \subseteq X_1$  is a subset of  $X_1$ ;
- (b) there exist the real constants  $l_1 \geq 0$ ,  $l_2 \geq 0$ ,  $l_3 \geq 0$  and  $p \in (0, 1]$  such that for every  $x, y, u \in D$  the following inequality holds:

$$\|[y, u; f] - [x, y; f]\| \leq l_1 \|x - u\|^p + l_2 \|x - y\|^p + l_3 \|y - u\|^p.$$

In [1] the hypothesis that the equation:

$$(1) \quad f(x) = 0$$

admits a simple solution  $x^*$  in adopted, and conditions for the convergence of the sequence  $(x_n)_{n \geq 0}$  generated by the chord method:

$$(2) \quad x_{n+1} = x_n - [x_{n-1}, x_n; f]^{-1} f(x_n), \quad n = 1, 2, \dots, \quad x_0, x_1 \in D_0$$

are given.

In a recent paper [2] there is shown that, with the hypotheses considered in [1], the convergence speed of the sequence generated by (2) and the error estimation are featured by the inequality:

$$(3) \quad \|x^* - x_{n+1}\| \leq \alpha d_0^{t_1^{n+1}},$$

where  $\alpha$  is a precised constant,  $0 < d_0 < 1$  and  $t_1$  is the positive root of the equation  $t^2 - t - p$ .

We shall admit further down that the divided difference operator fulfils the conditions (a) and (b), and search for supplementary conditions in order to make equation (1) admit a solution  $x^*$  into a precised domain  $D_0$  and the sequence  $(x_n)_{n \geq 0}$  generated by (2) converge to this solution.

Observe firstly that the identity:

$$(4) \quad x_n - [x_{n-1}, x_n; f]^{-1} f(x_n) = x_{n-1} - [x_{n-1}, x_n; f]^{-1} f(x_{n-1})$$

holds for every  $n = 1, 2, \dots$  with the hypothesis that the linear mapping  $[x_{n-1}, x_n; f]$  admits an inverse mapping.

The following identity

$$(5) \quad \begin{aligned} f(x_{n+1}) &= f(x_n) + [x_{n-1}, x_n; f](x_{n+1} - x_n) \\ &\quad + ([x_n, x_{n+1}; f] - [x_{n-1}, x_n; f])(x_{n+1} - x_n), \quad n = 1, 2, \dots \end{aligned}$$

Let  $B > 0, \alpha > 0, 0 < d_0 < 1$ , and  $x_0, x \in X_1$ . Consider the sphere

$$(6) \quad U = \left\{ x \in X_1 : \|x - x_0\| \leq \frac{B\alpha d_0}{1-d_0^{t_1-1}} \right\}$$

where  $t_1 = \frac{1+\sqrt{1+4p}}{2}$  that is, the positive root, the equation:

$$(7) \quad t^2 - t - p = 0$$

The following theorem holds:

**Theorem 1.** *If the divided difference  $[x, y; f]$  fulfils the conditions (a) and (b) for every  $x, y \in U$  and the following hypotheses:*

(1) *the mapping  $[x, y; f]$  admits a bounded inverse mapping for every  $x, y \in U$ , namely there exists a constant  $B > 0$  such that  $\|[x, y; f]^{-1}\| \leq B$*

(ii)

$$\alpha = \frac{1}{B^{(1+p)/p}(l_1+l_2+l_3)^{1/p}};$$

(iii)

$$\|x_1 - x_0\| \leq B\alpha d_0, \|f(x_0)\| \leq \alpha d_0, \|f(x_1)\| \leq \alpha d_0^{t_1}$$

*are also fulfilled, then equation (1) has at least one solution  $x^* \in U$  and the sequence  $(x_n)_{n \geq 0}$  generated by (2) converges to  $x^*$ , the convergence speed and the error estimation being featured by the inequality:*

$$\|x^* - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n(t_1-1)}}.$$

*Proof.* From (2) for  $n = 1$  we deduce:

$$\|x_2 - x_1\| \leq B \|f(x_1)\| \leq B\alpha d_0^{t_1}$$

from which, taking also into account iii. it follows

$$\begin{aligned} \|x_2 - x_0\| &\leq \|x_2 - x_1\| + \|x_1 - x_0\| \\ &\leq B\alpha d_0^{t_1} + B\alpha d_0 \\ &\leq B\alpha d_0 \left(1 + d_0^{t_1-1}\right) \\ &< \frac{B\alpha d_0}{1 - d_0^{t_1-1}} \end{aligned}$$

from which it results that  $x_2 \in U$ .

Using the fact that  $x_2 \in U$ , the identities (4) and (5), and the inequality a), we obtain

$$\begin{aligned} \|f(x_2)\| &\leq \|x_2 - x_1\| (l_1 \|x_2 - x_0\|^p + l_2 \|x_1 - x_0\|^p + l_3 \|x_2 - x_1\|^p) \\ &\leq B \|f(x_1)\| (l_1 B^p \|f(x_0)\|^p + l_2 \|x_1 - x_0\|^p + l_3 B^p \|f(x_1)\|^p) \\ &\leq B\alpha d_0^{t_1} (l_1 B^p \alpha^p d_0^p + l_2 B^p \alpha^p d_0^p + l_3 B^p \alpha^p d_0^{t_1}) \\ &\leq B^{p+1} \alpha^{p+1} d_0^{t_1+p} (l_1 + l_2 + l_3 d_0^{p(t_1-1)}) \\ &= B^{p+1} \alpha^{p+1} (l_1 + l_2 + l_3 d_0^{p(t_1-1)}) d_0^{t_1-p} \leq \alpha d_0^{t_1^2} \end{aligned}$$

since  $\alpha^p B^{p+1} (l_1 + l_2 + l_3 d_0^{p(t_1-1)}) \leq \alpha^p B^{p+1} (l_1 + l_2 + l_3) < 1$ .

From the above inequality follows therefore:

$$\|f(x_2)\| \leq \alpha d_0^{t_1^2}$$

Suppose by induction that:

- (a')  $x_i \in U$ ,  $i = 0, 1, \dots, k$ ;  
 (b')  $\|f(x_i)\| \leq \alpha d_0^{t_1^i}$ ,  $i = 1, 2, \dots, k$ .

Then, for  $x_{k+1}$  we have:

$$\begin{aligned}
 \|x_{k+1} - x_0\| &\leq \|x_{k+1} - x_k\| + \|x_k - x_{k-1}\| + \dots + \|x_1 - x_0\| \\
 &\leq B \|f(x_k)\| + B \|f(x_{k-1})\| + \dots + B\alpha d_0 \\
 &\leq B\alpha d_0^{t_1^k} + B\alpha d_0^{t_1^{k-1}} + \dots + B\alpha d_0 \\
 &= B\alpha d_0 \left(1 + d_0^{t_1-1} + d_0^{2(t_1-1)} + \dots + d_0^{t_1^k-1}\right) \\
 &\leq B\alpha d_0 \left(1 + d_0^{t_1-1} + d_0^{2(t_1-1)} + \dots + d_0^{k(t_1-1)}\right) \leq \frac{B\alpha d_0}{1-d_0^{t_1-1}}
 \end{aligned}$$

from which follows that  $x_{k+1} \in U$ . Proceeding now for  $x_{k+1}$ , as in the case of  $x_2$ , we obtain:

$$\begin{aligned}
 \|f(x_{k+1})\| &\leq B^{p+1} \alpha^{p+1} \left(l_1 + l_2 + l_3 d_0^{p t_1^{k-1} (t_1-1)}\right) d_0^{t_1^{k-1} (t_1+p)} \\
 &\leq B^{p+1} \alpha^{p+1} (l_1 + l_2 + l_3) d_0^{t_1^{k+1}} \leq \alpha d_0^{t_1^{k+1}}
 \end{aligned}$$

It results therefore that the relations (a') and (b') hold for every  $i \in \mathbb{N}$ .

Now we shall show that the sequence  $(x_n)_{n \geq 0}$  is fundamental.

Indeed, for every  $n, s \in \mathbb{N}$  we have:

$$\begin{aligned}
 \|x_{n+s} - x_n\| &\leq \sum_{k=n}^{n+s-1} \|x_{k+1} - x_k\| \leq \sum_{k=n}^{n+s-1} B \|f(x_k)\| \leq B\alpha \sum_{k=n}^{n+s-1} d_0^{t_1^k} \\
 &= B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^k - t_1^n} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^n (t_1^{k-n} - 1)} \\
 &\leq B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} d_0^{t_1^n (k-n)(t_1-1)} = B\alpha d_0^{t_1^n} \sum_{k=n}^{n+s-1} \left(d_0^{t_1^n (t_1-1)}\right)^{k-n} \\
 &\leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}}.
 \end{aligned}$$

By the last inequality and the fact that  $0 < d_0 < 1$  and  $t_1 > 1$  follows that the sequence  $(x_n)_{n \geq 2}$  is fundamental. For  $s \rightarrow \infty$ , from the inequality:

$$\|x_{n+s} - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}}$$

follows the inequality:

$$\|x^* - x_n\| \leq \frac{B\alpha d_0^{t_1^n}}{1-d_0^{t_1^n (t_1-1)}}.$$

In [1] Argyros showed that if the divided difference  $[x, y; f]$  fulfils the conditions (a) and (b) then  $f$  is Fréchet differentiable and  $[x, x; f] = f'(x)$ . From this fact follows that the mapping  $f$  is continuous on  $B$ : hence at limit for  $n \rightarrow \infty$  in the inequality:

$$\|f(x_n)\| \leq \alpha d_0^{t_1^n},$$

one obtains:

$$\|f(x^*)\| \leq 0$$

from which results  $f(x^*) = 0$ . With this the theorem is entirely proved  $\square$

**Remark 2.** In [5], [6] Schmidt imposes in the divided difference conditions similar to the conditions (a) and (b) given by Argyros in [1], but for  $p = 1$ . The same conditions are reproduced in [2], too.

## References

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