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ON COMPUTATIONAL COMPLEXITY IN SOLVING EQUATIONS BY STEFFENSEN-TYPE METHODS

ION PĂVĂLOIU
(Cluj-Napoca)

1. INTRODUCTION

This note is a continuation of the paper [4]. We shall establish here the optimal methods for the efficiency index of the class of Steffensen-type methods.

We adopt the efficiency index of an iterative process as being the number $I(\omega, p)$ given in [1] by:

$$I(\omega, p) = \omega^{\frac{1}{p}} \tag{1.1}$$

where ω is the convergence order of the iterative method and p represents the number of function evaluations that must be performed at each step. As it results from [1] and [4], the efficiency index can be defined as in (1.1) if we admit that the number of function evaluations is constant beginning from a certain step.

Let $I \subset R$ denote an interval of the real axis, and consider equation

$$f(x) = 0, \tag{1.2}$$

where $f : I \to R$. Suppose that equation (1.2) possesses a unique root $\bar{x} \in I$. Also suppose that f admits derivatives up to the order $m+1$, $m \in \mathbb{N}$, the $(m+1)$-th derivative of f is bounded on I, and $f'(x) \neq 0$ for all $x \in I$. If $F = f(I)$, then there exists the function $f^{-1} : F \to I$ and $\bar{x} = f^{-1}(0)$.

It is obvious that for approximating the solution of (1.2) it is sufficient to approximate f^{-1} at $y = 0$.

From the derivability hypotheses concerning f it follows that f^{-1} also possesses derivatives up to the order $m+1$, which are given by [2]:

$$\left[f^{-1}(y)\right]^{(k)} = \sum \tfrac{(2k-2-i_1)!(-1)^{k-1+i_1}}{i_2!i_3!\ldots i_k![f'(x)]^{2k-1}} \left(\tfrac{f'(x)}{1!}\right)^{i_1} \ldots \left(\tfrac{f^{(k)}(x)}{k!}\right)^{i_k} \tag{1.3}$$

$k=\overline{1,m+1}$, where the above sum extends over all the integer nonnegative solutions of the system:

$$\begin{aligned} i_2+2i_3+\ldots+(k-1)\,i_k &= k-1,\\ i_1+i_2+\ldots+i_k &= k-1. \end{aligned} \tag{1.4}$$

We shall consider the following general iterative process for solving the equation (1.2):

$$x_{n+k+1}-g\left(x_k,x_{k+1},\ldots,x_{k+n}\right),\qquad n\geq 0,\ k+1,2,\ldots, \tag{1.5}$$

where $g: I^{n+1}\to I$ is a function whose restriction to the diagonal of I^{n+1} coincides with a function $h: I\to I$, whose fixed point is $\bar{x}$, i.e. $g\left(x,x,\ldots,x\right)=h\left(x\right)$ for all $x\in I$ and $h\left(\bar{x}\right)=\bar{x}$.

In order to establish the optimal efficiency index of the class of Steffensen methods we shall adopt, as in [4], the following assumptions:

We consider as a function evaluation:

a) the evaluation of the function or of any of its derivatives at a certain point;

b) the evaluation by (1.3) of any of the derivatives of f^{-1} at a certain point;

c) the evaluation of g from (1.5) at a certain point.

2. GENERALIZED STEFFENSEN METHOD

Let

$$x_1,x_2,\ldots,x_{n+1} \tag{2.1}$$

be $n+1$ interpolation nodes from I and

$$y_1,y_2,\ldots,y_{n+1} \tag{2.2}$$

the values of f at $x_i, y_i=f\left(x_i\right),\ i=\overline{1,n+1}$.

Consider $n+1$ natural numbers $a_1,a_2,\ldots,a_{n+1}$ such that $a_i\geq 1\ i=\overline{1,n+1}$, and $a_1+a_2+\ldots+a_n+a_{n+1}=m+1$. Supposing that at each $x_i, i=\overline{1,n+1}$, we know that values of f and of its derivatives up to the order a_i-1, i.e. we know $f\left(x_i\right),f'\left(x_i\right),\ldots,f^{(a_i-1)}\left(x_i\right)$, by (1.3) we can get the values of f^{-1} and of its derivatives up to the order a_i-1.

We can now construct the Hermite inverse interpolation polynomial corresponding to f^{-1}, nodes (2.2), i.e. the following polynomial exists and is unique:

(2.3)
$$H\left(y_1,a_1;y_2,a_2;\ldots;y_{n+1},a_{n+1};f^{-1}|y\right)=$$
$$=\sum_{i=1}^{n+1}\sum_{j=0}^{a_i-1}\sum_{k=0}^{a_i-j-1}\left[f^{-1}\left(y_i\right)\right]^{(j)}\tfrac{1}{k!j!}\left[\tfrac{(y-y_i)^{a_i}}{\omega(y)}\right]^{(k)}_{y=y_i}\tfrac{\omega(y)}{(y-y_i)^{a_i-j-k}}$$

where

(2.4)
$$\omega\left(y\right)=\left(y-y_1\right)^{a_1}\left(y-y_2\right)^{a_2}\ldots\left(y-y_{n+1}\right)^{a_{n+1}}.$$

If x_{n+2} denotes the value of H at $y=0$ we have

(2.5)
$$\left|\bar{x}-x_{n+2}\right|\leq\tfrac{M_{m+1}}{(m+1)!}\left|f\left(x_1\right)\right|^{a_1}\left|f\left(x_2\right)\right|^{a_2}\ldots\left|f\left(x_{n+1}\right)\right|^{a_{n+1}},$$

where

$$M_{m+1}=\sup_{y\in F}\left|f^{-1}\left(y\right)^{(m+1)}\right|.$$

If $x_k,x_{k+1},\ldots,x_{k+n}\in I$ are $n+1$ approximations of $\bar{x}$, then a new approximation x_{k+n+1} can be obtained by (2.3):

(2.6)
$$x_{k+n+1}=H\left(y_k,a_1;y_{k+1},a_2;\ldots;y_{k+n},a_n;f^{-1}|0\right),\qquad k=1,2,\ldots,$$

with the error evaluation

(2.7)
$$\left|\bar{x}-x_{k+n+1}\right|\leq\tfrac{M_{m+1}}{(m+1)!}\left|f\left(x_k\right)\right|^{a_1}\left|f\left(x_{k+1}\right)\right|^{a_2}\ldots\left|f\left(x_{k+n}\right)\right|^{a_{n+1}}.$$

Method (2.6) is called Hermite-like iterative method.

Consider a function $\varphi:I\to I$ whose fixed point from I is $\bar{x}$ i.e. $\varphi\left(\bar{x}\right)=\bar{x}$, and suppose there exists a real number $\alpha>0$ such that

(2.8)
$$\left|f\left(\varphi\left(x\right)\right)\right|\leq\alpha\left|f\left(x\right)\right|,\qquad\text{for all } x\in I.$$

Let $\varphi_1\left(x\right)=\varphi\left(x\right),\varphi_2\left(x\right)=\varphi\left(\varphi_1\left(x\right)\right),\ \varphi_3\left(x\right)=\varphi\left(\varphi_2\left(x\right)\right),\ldots,\varphi_n\left(x\right)=\varphi\left(\varphi_{n-1}\left(x\right)\right)$, be the iterations up to the order n of the function φ.

To increase the convergence order of method (2.6) we can do as if follows.

Let $x_k\in I$ be a certain approximation of the solution $\bar{x}$ of equation (1.2) and $u_k=x_k,u_{k+1}=\varphi_1\left(x_k\right),\ldots,u_{n+k}=\varphi_n\left(x_k\right)$. Consider the values $\bar{y}_i=f\left(u_i\right)i=\overline{k,n+k}$ as interpolation nodes in (2.3). Then x_{k+1}, the next approximation of $\bar{x}$, is given by:

(2.9)
$$x_{k+1}=H\left(\bar{y}_k,a_1;\bar{y}_{k+1},a_2,\ldots,\bar{y}_{k+n},a_{n+1};f^{-1}|0\right).$$

Repeating this process, called Steffensen type iterative method, we obtain a sequence $\left(x_n\right)_{n\geq0}$ of approximations of $\bar{x}$.

Using (2.8) and (2.7) it can be easily seen that the convergence order of (2.9) is $m+1$.

3. THE EFFICIENCY INDEX OF STEFFENSEN-TYPE METHODS

As it can be seen above, at each iteration step in (2.9) we have the following function evaluations:

1) n values of φ to obtain the interpolation nodes $u_{k+i}, i=\overline{1,n}$;

2) $n+1$ values of f at the nodes $u_{k+i}, i=\overline{0,n}$;

3) at each interpolation node $u_{k+i}, i=\overline{0,n}$ we compute the values of successive derivatives of f up to the order $a_{i+1}-1$, altogether $m-n$ function evaluations:

4) by (1.3) we evaluate the successive derivatives of f^{-1} at $\bar{y}_{k+1}=f\left(u_{k+i}\right),\ i=\overline{0,n}$ up to the order $a_{i+1}-1$, altogether $m-n$ function evaluations;

5) finally, consider (2.9) as a single function evaluation.

Summing up, we obtain altogether $2\left(m+1\right)$ function evaluations.

Using (1.1) we obtain the following expression for the efficiency index of the class of Steffensen-type methods:

$$I\left(m+1,2\left(m+1\right)\right)=\left(m+1\right)^{\frac{1}{2(m+1)}} \tag{3.1}$$

Elementary considerations on the behaviour of the function $h:(0,\infty)\to\mathbb{R},\ h\left(t\right)=t^{\frac{1}{2t}}$ lead us to the conclusion that the function $I\left(m+1,2\left(m+1\right)\right)$ attains its maximum at $m=2$.

Note that the efficiency index (3.1) does not depend on the number of interpolation nodes.

From $m=2$ and $a_1+a_2+\ldots+a_{n+1}=m+1,\ a_i\geq 1\ i=\overline{1,n+1}$ it follows that $n\leq 2$.

We shall successively analyse all the cases that lead us to the optimal methods form (2.9).

A. $a_1+a_2+a_3=3$, i.e. $a_1=a_2=a_3=1$. Then (2.3) becomes the Lagrange's inverse interpolation polynomial, and (2.9) is written:

$$\begin{aligned} x_{k+1}=&x_k-\frac{f\left(x_k\right)}{\left[x_k,\varphi\left(x_k\right);f\right]}\\ &-\frac{\left[x_k,\varphi\left(x_k\right),\varphi\left(\varphi\left(x_k\right)\right);f\right]f\left(x_k\right)f\left(\varphi\left(x_k\right)\right)}{\left[x_k,\varphi\left(x_k\right);f\right]\left[x_k,\varphi\left(\varphi\left(x_k\right)\right);f\right]\left[\varphi\left(x_k\right),\varphi\left(\varphi\left(x_k\right)\right);f\right]} \end{aligned} \tag{3.2}$$

$x_0\in I,\ k=0,1,\ldots,$ where $[u,v;f]$ respectively $[u,v,w;f]$ denote the first, respectively the second order divided differences of f.

B. $a_1 + a_2 = 3$, i.e. $a_1 = 2$, $a_2 = 1$ or $a_1 = 1$ and $a_2 = 2$. When $a_1 = 2$, $a_2 = 1$ we obtain the following method:

$$x_{k+1} = x_k = -\frac{f(x_k)}{f'(x_k)} + \frac{(f'(x_k) - [x_k, \varphi(x_k); f])\, f^2(x_k)}{[x_k, \varphi(x_k); f]^2 \cdot f'(x_k)(\varphi(x_k) - x_k)} \tag{3.3}$$

$x_0, \in I,\ k = 0, 1, \ldots$, and when $a_1 = 1,\ a_2 = 2$ it follows:

$$x_{k+1} = \varphi(x_k) - \frac{f(\varphi(x_k))}{f'(\varphi(x_k))} + \frac{([x_k, \varphi(x_k); f] - f'(\varphi(x_k)))\, f^2(\varphi(x_k))}{[x_k, \varphi(x_k); f]^2\, f'(\varphi(x_k))(\varphi(x_k) - x_k)}, \tag{3.4}$$

$x_0 \in I,\ k = 0, 1, \ldots$

C. $a_1 = 3$. In this case we get from (2.9) the third order Chebyshev iterative method, studied in [4].

In conclusion, the following theorem holds:

THEOREM 3.1. *Under the assumptions* a)–c) *from* 1., *in the class of Steffensen-type iterative methods any of the methods* (3.2), (3.3) *or* (3.4) *is optimal, i.e. has the greatest efficiency index.*

REMARK. For the particular case when $a_1 = a_2 = \ldots = a_{n+1} = q$ the condition of optimality for the efficiency index gives us two possibilities, namely $q = 3,\ n = 0$, hence the case C. or $q = 1,\ n = 2$, hence the case A. □

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Academia Română
Institutul de Calcul
"Tiberiu Popoviciu"
P.O. Box 68
3400 Cluj-Napoca 1
România