

ON THE HERON'S METHOD FOR APPROXIMATING
THE CUBIC ROOT OF A REAL NUMBER

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1. INTRODUCTION

In this paper we shall specify and go deeply into some problems presented in [1], concerning the Heron's method for approximating the cubic root of a positive number.

The authors of [1] construct a method based on the Heron's algorithm for computing the cubic root of 100.

The method works as follows: Given two real numbers a and b satisfying $a^3 < N < b^3$, the Heron's method for approximating $\sqrt[3]{N}$ is

$$(1.1) \quad \phi(N, a, b) = a + \frac{bd_1}{bd_1 + ad_2} (b - a),$$

where $d_1 = N - a^3$ and $d_2 = b^3 - N$. We shall show that the approximation (1.1) of $\sqrt[3]{N}$ follows from the *regula falsi* applied to the equation $x^2 - \frac{N}{x} = 0$ [2]. This will give a rigorous interpretation of (1.1), and the results from [1] will be reached again.

Using results from [4], we shall give other error bounds than those in [1]. On the other hand, the method is generalized to the case $\sqrt[p]{N}$, $p \in \mathbb{N}$, $p \geq 2$, the method also offering bilateral approximations. Some remarks on applying the results in [4] for the error bounds will lead us to the generalization of the Heron's method.

2. HERONS'S METHOD AND REGULA FALSI

A. In order to approximate the cubic root of $N > 0$ by (1.1), consider the function $f : [a, b] \rightarrow \mathbb{R}$, $f(x) = x^3 - N$, $0 < a < b$ and the function $g : [a, b] \rightarrow \mathbb{R}$, $g(x) = \frac{f(x)}{x}$.

It is well known that *regula falsi* applied to the equation $g(x) = 0$ leads to the following approximation of its root

$$(2.1) \quad c = a - \frac{g(a)}{[a, b; g]},$$

$[u, v; g]$ denoting the first-order divided difference of g on the nodes u and v . It can be easily verified that $c = \phi(N, a, b)$.

Taking into account that $c \in (a, b)$ and denoting by $[u, v, w; g]$ the second-order divided difference of g on the points u, v, w , we get

$$(2.2) \quad g(x) = g(a) + [a, b; g](x - a) + [a, b, x; g](x - a)(x - b)$$

for all $x \in (a, b)$.

For $x = \sqrt[3]{N}$ in (2.2) we obtain

$$g(a) + [a, b; g](\sqrt[3]{N} - a) + [a, b, \sqrt[3]{N}; g](\sqrt[3]{N} - a)(\sqrt[3]{N} - b) = 0,$$

from which, by dividing $[a, b; g]$ it follows

$$(2.3) \quad c - \sqrt[3]{N} = \frac{[a, b, \sqrt[3]{N}; g]}{[a, b; g]}(\sqrt[3]{N} - a)(\sqrt[3]{N} - b).$$

An elementary calculation on $\frac{[a, b, \sqrt[3]{N}; g]}{[a, b; g]}$ shows that

$$(2.4) \quad \frac{c - \sqrt[3]{N}}{\sqrt[3]{N}} = \frac{\sqrt[3]{N} + \sqrt{ab}}{\sqrt[3]{N}[ab(a+b)+N]}(\sqrt[3]{N} - a)(\sqrt[3]{N} - b)(\sqrt[3]{N} - \sqrt{ab}),$$

which gives Theorem 3, [1].

Taking into account the above remarks and using the evaluations obtained by T. Popoviciu in [4], (2.3) gives the following error bounds

$$(2.5) \quad \frac{m_2}{M_1}(\sqrt[3]{N} - a)(b - \sqrt[3]{N}) \leq |c - \sqrt[3]{N}| \leq \sqrt{\frac{M_2}{m_1}}(\sqrt[3]{N} - a)(b - \sqrt[3]{N}),$$

where

$$\begin{aligned}
m_1 &= 3a \\
m_2 &= \min \left\{ \frac{N}{a^3} - 1, 1 - \frac{N}{b^3} \right\} \\
M_1 &= \max \left\{ \frac{2a^3 + N}{a^2}, \frac{2b^3 + N}{b^2} \right\} \\
M_2 &= \max \left\{ \frac{N}{a^3} - 1, 1 - \frac{N}{b^3} \right\}.
\end{aligned}$$

Note that (2.5) leads to a very good error evaluation; since a and b are close to N , then m_2 and M_2 are close to zero. This is implied by the fact that the function g and its second derivative vanish at the same point $x = \sqrt[3]{N}$.

B. It can be easily seen that the method presented at **A** can be generalized. For the approximation of the root of order p of the real number N , $p \in \mathbb{N}$, $p \geq 2$, consider the function $f_1 : [a_1, b_1] \rightarrow \mathbb{R}$,

$$f_1(x) = x^p - N, \quad 0 < a_1 < b_1, \quad a_1^p < N < b_1^p$$

and the function $g_1 : [a_1, b_1] \rightarrow \mathbb{R}$,

$$g_1(x) = \frac{f_1(x)}{x^q}, \quad \text{where } q = \frac{p-1}{2}.$$

The function g_1 satisfies $g_1(\sqrt[p]{N}) = g_1''(\sqrt[p]{N})$.

Applying *regula falsi* to the equation $g_1(x) = 0$, we obtain

$$(2.6) \quad c_1 = a_1 - \frac{g_1(a_1)}{[a_1, b_1; g_1]}$$

Similarly to **A**, we obtain

$$(2.7) \quad c_1 - \sqrt[p]{N} = \frac{[a_1, b_1, \sqrt[p]{N}; g_1]}{[a_1, b_1; g_1]} (\sqrt[p]{N} - b_1) (\sqrt[p]{N} - a_1),$$

which gives

$$(2.8) \quad \frac{t_2}{2t_1} (\sqrt[p]{N} - a_1) (b_1 - \sqrt[p]{N}) \leq |c_1 - \sqrt[p]{N}| \leq \frac{T_2}{2t_1} (\sqrt[p]{N} - a_1) (b_1 - \sqrt[p]{N}),$$

where

$$\begin{aligned}
t_1 &= p a_1^{\frac{p-1}{2}} \\
t_2 &= \min \left\{ \frac{(p-1)(p+1)(N - a_1^p)}{4a_1^{\frac{p+3}{2}}}, \frac{(p-1)(p+1)(b_1^p - N)}{4b_1^{\frac{p+3}{2}}} \right\}
\end{aligned}$$

$$T_1 = \max \left\{ \frac{(p+1)a_1^p + (p-1)N}{2a_1^{\frac{p+1}{2}}}; \frac{(p+1)b_1^p + (b-1)N}{2b_1^{\frac{p+1}{2}}} \right\}$$

$$T_2 = \max \left\{ \frac{(p-1)(p+1)(N-a_1^p)}{4a_1^{\frac{p+3}{2}}}; \frac{(p-1)(p+1)(b_1^p - N)}{4b_1^{\frac{p+3}{2}}} \right\}.$$

3. STEFFENSEN'S METHOD FOR APPROXIMATING THE p TH-ORDER ROOT

Let $I = [\alpha, \beta]$, $\alpha < \beta$ be an interval of the real axis.

Consider the equation

$$(3.1) \quad F(x) = 0,$$

where $F : I \rightarrow \mathbb{R}$. Suppose that equation (3.1) has a root $\bar{x} \in (\alpha, \beta)$. Consider also a function $h : I \rightarrow \mathbb{R}$ such that equation

$$(3.2) \quad x - h(x) = 0$$

is equivalent to (3.1).

The Steffensen's method consists in the generation of two sequences (x_n) and $(h(x_n))$ by the relations

$$(3.3) \quad x_{n+1} = x_n - \frac{F(x_n)}{[x_n, h(x_n); F]}, \quad n = 0, 1, \dots, \quad x_0 \in I.$$

As we shall see, this method offers the possibility to obtain better both upper and lower approximations, by starting with a lower approximation of $\sqrt[p]{N}$. Then, by applying only once the *regula falsi* (2.6), the precision can be increased.

As concerns the convergence of (x_n) and $(h(x_n))$ in (3.3), in [3] the following theorem is proved.

THEOREM 3.1. [3]. *If the functions $F : I \rightarrow \mathbb{R}$ and $h : I \rightarrow \mathbb{R}$ are continuous and satisfy the following conditions:*

- i) *the function h is decreasing on I ,*
- ii) *the function F is increasing and convex on I ,*
- iii) *there exists $x_0 \in I$ such that $F(x_0) < 0$ and $h(x_0) \in I$,*
- iv) *the equations (3.1) and (3.2) are equivalent,*

then the following properties hold:

- j) the sequence (x_n) is increasing,
- jj) the sequence $(h(x_n))$ is decreasing,
- jjj) $\lim x_n = \lim h(x_n) = \bar{x}$
- jjv) the relations $x_n \leq x_{n+1} \leq \bar{x} \leq h(x_n)$ hold for all $n = 0, 1, \dots$,
- v) $\bar{x} - x_{n+1} < h(x_n) - x_{n+1}$.

Applying this Theorem for $F : [\alpha, \beta] \rightarrow \mathbb{R}$, $F(x) = x^p - N$, $h : [\alpha, \beta] \rightarrow \mathbb{R}$,

$$h(x) = x - \frac{F(x)}{p\alpha^{p-1}},$$

where $0 < \alpha < \beta$ and $p \in \mathbb{N}$, $p \geq 2$, we obtain

$$x_{n+1} = x_n + \frac{(p\alpha^{p-1})^{p-1}(x_n - N)^2}{(p\alpha^{p-1}x_n - x_n^p + N)^p - (p\alpha^{p-1}x_n)^p}, \quad n = 0, 1, \dots, x_0 = \alpha; \alpha^p < N.$$

Since F is increasing and convex $[\alpha, \beta]$, it follows that h is decreasing on $[\alpha, \beta]$, and the equations $F(x) = 0$ and $h(x) - x = 0$ are equivalent. So the conclusion of Theorem 3.1 follows.

The sequences (x_n) and $(h(x_n))$ being convergent, it follows that for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ we have

$$h(x_n) - x_n < \varepsilon,$$

which implies $\sqrt[p]{N} - x_n < \varepsilon$ and $h(x_n) - \sqrt[p]{N} < \varepsilon$.

If we use (2.6) for $a_1 = x_n$ and $b_1 = h(x_n)$ and $g_1(x) = \frac{F(x)}{x^{\frac{p-1}{2}}}$, and we denote the approximation obtained by c_n , then by (2.8) we have

$$\left| c_n - \sqrt[p]{N} \right| \leq \frac{T'_2}{2t'_1} \varepsilon^2,$$

where

$$\left\{ \begin{array}{l} T'_2 = \max \left\{ \frac{(p-1)(p+1)(N-x_n^p)}{4x_n^{\frac{p+3}{2}}}, \frac{(p-1)(p+1)(h^p(x_n)-N)}{4h(x_n)^{\frac{p+3}{2}}} \right\} \\ t'_1 = px_n^{\frac{p-1}{2}} \end{array} \right. .$$

4. A NUMERICAL EXAMPLE

We intend to apply the method described in Section 3 for the approximation of the number $\sqrt[5]{100}$, i.e., for solving the equation $x^5 - 100 = 0$. In this case we have

$$F(x) = x^5 - 100$$

and, taking $\alpha = 2$, for the function h we have

$$h(x) = x - \frac{1}{80}F(x).$$

Considering $x_0 = \alpha = 2$ and using (3.3), with F and h given above, we obtain for the sequences $(x_n)_{n \geq 0}$ and $(h(x_n))_{n \geq 0}$ the following values:

| n | x_n | $h(x_n)$ | $\varepsilon_n = h(x_n) - x_n$ |
|-----|-----------------|-----------------|------------------------------------|
| 0 | 2.000 000 000 0 | 2.850 000 000 0 | $8.500 000 000 0 \cdot 10^{-01}$ |
| 1 | 2.370 444 507 2 | 2.684 911 796 6 | $3.144 672 894 1 \cdot 10^{-01}$ |
| 2 | 2.492 753 689 2 | 2.539 639 492 8 | $4.688 580 357 8 \cdot 10^{-02}$ |
| 3 | 2.511 465 149 3 | 2.512 513 019 4 | $1.047 870 071 5 \cdot 10^{-03}$ |
| 4 | 2.511 886 221 3 | 2.511 886 744 3 | $5.229 121 597 9 \cdot 10^{-07}$ |
| 5 | 2.511 886 431 5 | 2.511 886 431 5 | $3.637 978 807 1 \cdot 10^{-12}$. |

Table 4.1.

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